Waseda University Institute of Finance

Working Paper Series

WIF-05-004

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http://www.waseda.jp/wnfs/nif/index.html

Option Pricing with Hidden Markov Models *

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May 6 & September 22, 2005

Abstract

In this paper, we derive an analytic formula for pricing European call options under the setting of discrete-time Hidden Markov Models (HMM). HMM is specified by a state equation with the time-homogeneous transition probability matrix and an observation equation which describes asset prices by the log-normal model in which both drift and volatility parameters switch according to the state. With the setup above, we derive an analytic formula for pricing European call option. When compared to the existing option pricing models which characterize stochastic volatility in asset prices, the advantages of the formula are: (1) it is an analytic formula, (2) easy to interpret its meanings and, (3) able to capture the persistence of volatility in the risky asset prices. We also implement some empirical analyses to show that HMM is able to express so-called volatility smiles.

Keywords: option pricing, Hidden Markov Models, Baum-Welch algorithm, volatility smile

1 Introduction

The option pricing theory has been intensively studied since the pioneering works of Black-Scholes (1973) and Merton (1973). In Black-Scholes model, the underlying asset price process is described by geometric Brownian motion in which the drift and volatility

*Presented/will be presented under the same title at 2005 Daiwa International Workshop on Financial Engineering, 大手町サンケイプラザ, July 22, 2005; 日本金融・証券計量・工学学会 (JAFEE) 2005 夏季大会, 慶應義塾大学 三田キャンパス, 2005 年 8 月 5 日; Quantitative Methods in Finance 2005 Conference, Manly Pacific Sydney Hotel, Dec. 14-17, 2005.

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parameters are assumed to be deterministic. It is known, however, that the volatility in asset price processes in the financial market would depend on the past information. For example, the persistence in the level of volatility is frequently observed. That is, the days with high volatility will follow the days with high volatility and, the days with low volatility will follow the days with low volatility. For these reasons, the implied volatility, which is obtained when the market price of European call option is equated with the Black-Scholes model, is not constant but varying with respect to the time to maturity and strike price of option. This phenomenon is known as volatility smile. The several time-series models, such as ARCH (Engle, 1982) and GARCH (Bollerslev, 1986) models, have been proposed to characterize the volatility dynamics. Also, Hull and White (1987) and Heston (1993) introduce stochastic volatility models . Sharing these issues involved in volatility modeling, we derive an analytic formula for pricing European call option when asset price processes are subject to hidden Markov models (HMM). Also we carry out empirical analyses to validate the model.

The study of hidden Markov models started in the late of 1960s by seminal works of Baum and Petrie (1966), Baum and Eagon (1967), Baum et al (1970), Baum (1972), and many others. Since then, the models are applied in many research fields such as cognitive science and biological science. Concerning the literature of time-series analysis in econometrics, HMM is introduced by Hamilton (1989) as regime switching models. The model is widely used in detecting structural breaks or turning points in economic time-series (Hamilton, 1994). Timmermann (2000) shows that the model is able to express the wide range of moments in asset prices. Concerning the option pricing with HMM, Elliott and Buffington (2002) express an option price utilizing the characteristic function in continuous-time framework. Duan et al (2002) derive an analytic European call option price with two state HMM.

In this paper, sharing the heart of the preceding papers, we address three issues. Firstly, we derive an analytic formula for pricing European call option under the setting of discrete-time HMM. Secondly, we estimate the model by Baum-Welch algorithm with scaling in computing forward-backward probabilities. Thirdly, we implement some empirical analyses. We estimated the parameters in HMM from the Japanese financial market data. We then compute the European call option price and implied volatility with our model, in comparison with the Black-Scholes model.

The paper is organized as follows. In section 2, we derive an analytic formula for pricing European call option under the setting of discrete-time HMM. In section 3, we implement empirical analyses to apply the model. In section 4, we conclude.

2 Model

We consider a market where a risk-free asset, a risky asset, and a European call option written on the underlying risky asset are being traded. It is assumed that N economic states exist in the market at discrete time t ($t = 0, 1, \ldots, T$). We denote the state as $Y \triangleq \{Y_t\ ;\ t = 0, \ldots, T\}$ and write $\mathcal{F}_t^Y \triangleq \sigma(Y_0, Y_1, \ldots, Y_t)$. The state space of Y_t is $\{e_1, \ldots, e_i, \ldots, e_N\}$ where $e_i \in \mathbf{R}^N$ ($i = 1, \ldots, N$). Here the i-th element of e_i is 1 and otherwise 0.

We assume that the state Y_t follows a first-order Markov process with the homogeneous transition probability as follows:

$$\mathbf{P} \;=\; (p_{ji})_{1\le i,j\le N} = (\, \Pr(Y_{t+1} = e_j | Y_t = e_i)\,)_{1\le i,j\le N}\ . \tag{2.1}$$

Here, p_{ji} satisfies:

$$p_{ji} \ge 0\ (i, j = 1, \ldots, N), \quad \sum_{j=1}^{K} p_{ji} = 1\ (i = 1, \ldots, N)\ .$$

Also we define the initial state probability as

$$\boldsymbol{\pi} \triangleq (\pi_i = \Pr(Y_0 = e_i))'_{1\le i\le N}\ , \tag{2.2}$$

where the superscript $'$ represents the transpose. Now, the state Y_t can be expressed by a state equation as follows:

$$Y_{t+1} \;=\; \mathbf{P}Y_t + M_{t+1}\ , \tag{2.3}$$

Where M_{t+1} is a $\mathcal{F}_t^Y$-martingale increment.

On the other hand, when the state Y_t at time t is given, the log-return of the risky asset is assumed to be described as an observation equation:

$$\log\left(\frac{S_t}{S_{t-1}}\right)\Bigg| Y_t = \mu(Y_t) - \frac{1}{2}\sigma^2(Y_t) + \sigma(Y_t)\varepsilon_t\ , \tag{2.4}$$

where $\varepsilon_t \underset{\text{I.I.D.}}{\sim} N(0,1)$ indicates mutually independent, identical standard normal error term under the probability measure P. Also, we write $\mathcal{F}_t^R \triangleq \sigma(\log(S_1/S_0), \ldots, \log(S_t/S_{t-1}))$ and $\mathcal{F}_t^{R,Y} \triangleq \{\mathcal{F}_t^R, \mathcal{F}_t^Y\}$.

Here the drift and diffusion parameters are assumed to take values corresponding to the state. That is, with the notation

$$\boldsymbol{\mu} = (\mu_1 \ldots \mu_i \ldots \mu_N)'\ ,\ \boldsymbol{\sigma} = (\sigma_1 \ldots \sigma_i \ldots \sigma_N)'\ ,$$

the drift and diffusion parameters in the period t are assumed to be:

$$\mu(Y_t) = \langle \boldsymbol{\mu},\ Y_t \rangle \ , \tag{2.5}$$

$$\sigma(Y_t) = \langle \boldsymbol{\sigma},\ Y_t \rangle \ . \tag{2.6}$$

Where the operator $\langle\ ,\ \rangle$ indicates the inner product. The equation above signifies that the drift and diffusion parameters which characterize the return and risk in the asset log-return switch from period to period depending on the state.

We also assume that the log-return of the risk-free asset, given the state Y_t, is

$$\begin{aligned} r(Y_t) &= \langle \boldsymbol{r},\ Y_t \rangle \text{ with,} \\ \boldsymbol{r} &= (r_1 \ldots r_i \ldots r_N)' . \end{aligned} \tag{2.7}$$

This shows that the risk-free rate takes the value corresponding to the state, too. By the Locally Risk-Neutral Valuation Relationship of Duan (1995) and Duan et al (2002), under the equilibrium probability measure Q, the asset price process of Eq. (2.4) becomes

$$\log\left(\frac{S_t}{S_{t-1}}\right)\Bigg| Y_t = r_t - \frac{1}{2}\sigma^2(Y_t) + \sigma(Y_t)\epsilon_t^* \ . \tag{2.8}$$

Here $\epsilon_t^* \sim N(0,1)$ under the measure Q.

If all the states were observable, that is, Y_t $(t = 1, \cdots, T)$ were given, the call option price would be

$$C(0,T,S_0,\mathcal{F}_T^Y) = E^Q\left[\exp\left(-\sum_{t=1}^{T} r(Y_t)\right)\max(S_T - K, 0)\right] \ ,$$

under the measure Q. From Eq. (2.8), we obtain the call option price as follows:

$$C(0,T,S_0,\mathscr{F}_T^Y) = S_0\Phi(d_1) - e^{-\sum_{t=1}^{T} r(Y_t)} K\Phi(d_2) \ .$$

Here d_1 and d_2 are given as

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + \sum_{t=1}^{T}\left(r(Y_t) + \frac{1}{2}\sigma^2(Y_t)\right)}{\sqrt{\sum_{t=1}^{T}\sigma^2(Y_t)}} \ , \\ d_2 &= d_1 - \sqrt{\sum_{t=1}^{T}\sigma^2(Y_t)} \ . \end{aligned}$$

In the market, however, the economic state is unobservable. When the economic state is hidden, we should consider the trajectory of the state transition until the maturity T. Denote the occupation time in the state e_i $(i = 1, \ldots, N)$, from time 0 to time t, as

$$O_T^i \triangleq \sum_{t=1}^{T} \langle Y_t, e_i \rangle$$

Then the summations of Eqs. (2.6) and (2.7) can be rewritten as

$$\begin{aligned}
\sum_{t=1}^{T} r(Y_t) &= \sum_{t=1}^{T} \langle \boldsymbol{r}, Y_t \rangle = \sum_{i=1}^{N} r_i O_T^i = \sum_{i=1}^{N-1} r_i O_T^i + r_N \left(T - \sum_{i=1}^{N-1} O_T^i \right) , \\
\sum_{t=1}^{T} \sigma^2(Y_t) &= \sum_{t=1}^{T} \langle \boldsymbol{\sigma}^2, Y_t \rangle = \sum_{i=1}^{N} \sigma_i^2 O_T^i = \sum_{i=1}^{N-1} \sigma_i^2 O_T^i + \sigma_N^2 \left(T - \sum_{i=1}^{N-1} O_T^i \right) .
\end{aligned}$$

Hence take another expectation with respect to the joint probability of the occupation-time to obtain the European call option price with Hidden Markov Models of Eqs. (2.3) and (2.8):

$$
\begin{aligned}
&C(0,T,S_0) && (2.9)\\
&= \sum_{\tau_1=0}^{T}\cdots\sum_{\tau_{N-1}=0}^{T} \Pr(O_T^1=\tau_1,\cdots,O_T^{N-1}=\tau_{N-1})\left[S_0\Phi(d_1)-e^{-\sum_{i=1}^{N-1}r_i\tau_i-r_N(T-\sum_{i=1}^{N-1}\tau_i)}K\Phi(d_2)\right] &&\\
&= \sum_{j=1}^{N}\pi_j C(0,T,S_0,Y_0=e_j)\ , && (2.10)
\end{aligned}
$$

where π_j is defined in Eq. (2.2) and, the call option price conditioned on the initial state is defined as

$$
\begin{aligned}
C(0,T,S_0,Y_0=e_j) \ \triangleq\ & \sum_{\tau_1=0}^{T}\cdots\sum_{\tau_{N-1}=0}^{T} \Pr(O_T^1=\tau_1,\cdots,O_T^{N-1}=\tau_{N-1}|Y_0=e_j)\\
& \times\left[S_0\Phi(d_1)-e^{-\sum_{i=1}^{N-1}r_i\tau_i-r_N(T-\sum_{i=1}^{N-1}\tau_i)}K\Phi(d_2)\right]\ . \qquad (2.11)
\end{aligned}
$$

Also, d_1 and d_2 are given as follows

$$
\begin{aligned}
d_1 &= \frac{\log(S_0/K)+\sum_{i=1}^{N-1}\left(r_i+\frac{1}{2}\sigma_i^2\right)\tau_i+(r_N+\frac{1}{2}\sigma_N^2)(T-\sum_{i=1}^{N-1}\tau_i)}{\sqrt{\sum_{i=1}^{N-1}\sigma_i^2\tau_i+\sigma_N^2\left(T-\sum_{i=1}^{N-1}\tau_i\right)}}\ ,\\
d_2 &= d_1-\sqrt{\sum_{i=1}^{N-1}\sigma_i^2\tau_i+\sigma_N^2\left(T-\sum_{i=1}^{N-1}\tau_i\right)}\ .
\end{aligned}
$$

As a special case of $N=1$, one can easily see that the Eq. (2.10) corresponds to the Black-Scholes formula.

How to calculate the European call option price based on N-state HMM? Firstly, one has to prepare input. Besides the initial underlying asset price of S_0, the strike price of K, and the time to maturity of T, one needs to estimate the model parameters of $r_i,\ \sigma_i,\ \pi_i,\ p_{ji}(i,j=1,\dots,N)$. We estimate HMM by Baum-Welch algorithm with scaling in computing forward-backward probabilities to avoid underflow in computation.

Secondly, one has to prepare two functions besides the four rules of arithmetic. The one is the standard normal distribution function, $\Phi(\cdot)$, which is already provided in most software. The other is the joint probability of occupation-time, $\Pr(O_T^1=\tau_1,\cdots,O_T^{N-1}=\tau_{N-1})$. This can be obtained in the following recursive equations.

The joint probability of occupation-time conditioned on the initial state can be written as

$$
\Pr(O_T^1=\tau_1,O_T^2=\tau_2,\cdots,O_T^{N-1}=\tau_{N-1}|Y_0=e_j)
$$

$$
\begin{aligned}
&= \Pr(O_T^1 = \tau_1 | Y_0 = e_j) \cdot \Pr(O_T^2 = \tau_2 | O_T^1 = \tau_1, Y_0 = e_j) \cdot \ \ldots \\
&\cdots \ \cdot \Pr(O_T^{N-1} = \tau_{N-1} | O_T^1 = \tau_1, O_T^2 = \tau_2, \cdots, O_T^{N-2} = \tau_{N-2}, Y_0 = e_j) \ . \qquad (2.12)
\end{aligned}
$$

where,

$$
\begin{aligned}
&\Pr(O_T^2 = \tau_2 | O_T^1 = \tau_1, Y_0 = e_j) \\
&= \begin{cases} \Pr(O_{T-\tau_1}^2 = \tau_2 | Y_0 = e_j) & (\text{if } 0 \leq \tau_2 \leq T - \tau_1) \\ 0 & (\text{otherwise}) \ , \end{cases} \\
&\vdots \\
&\Pr(O_T^{N-1} = \tau_{N-1} | O_T^1 = \tau_1, O_T^2 = \tau_2, \cdots, O_T^{N-2} = \tau_{N-2}, Y_0 = e_j) \\
&= \begin{cases} \Pr\left(O_{T-\sum_{i=1}^{N-2} \tau_i}^{N-1} = \tau_{N-1} \middle| Y_0 = e_j \right) & (\text{if } 0 \leq \tau_{N-1} \leq T - \sum_{i=1}^{N-2} \tau_i) \\ 0 & (\text{otherwise}) \ . \end{cases}
\end{aligned}
$$

To compute the above joint probability, we need to know the occupation-time probabilities for each state. These probabilities can be computed recursively by the following relations.

Write the probability from state i to state j in u times transition as

$$
\mathbf{P}(u) = \Big(p_{ji}(u)\Big)_{1 \leq i,j \leq N} = \Big(\Pr\left(Y_{t+u} = e_j | Y_t = e_i\right) \Big)_{1 \leq i,j \leq N} \ (u = 1, \ldots, T) \ . \qquad (2.13)
$$

From the Chapman-Kolmogorov equation, we have

$$
\mathbf{P}(u) = \mathbf{P}^u \ (u = 1, \ldots, T) \ , \qquad (2.14)
$$

where $\mathbf{P}$ is the transition probability. Define the first passage probability as

$$
f_{ji}(t) = \Pr\left(Y_t = e_j, Y_{t-1} \neq e_j, \ldots, Y_1 \neq e_j | Y_0 = e_i\right) \ (i, j = 1, \ldots, N; \ t = 1, \ldots, T) \ (2.15)
$$

With Eq. (2.14), the first passage probability, $f_{ji}(t)$ can be computed in the recursive equations:

$$
\begin{aligned}
f_{ji}(t) &= p_{ji}(t) - \sum_{k=1}^{t-1} p_{jj}(t-k) f_{ji}(k) \ (t = 2, \ldots, T) \ , & (2.16) \\
f_{ji}(1) &= \Pr(Y_1 = e_j | Y_0 = e_i) = p_{ji} \ . & (2.17)
\end{aligned}
$$

Given the initial state $Y_0 = e_j$, we obtain the recursive equations concerning the occupation-time probability in state i ($i = 1, \ldots, N$).

$$
\Pr(O_1^i = 1 | Y_0 = e_j) = \Pr(Y_1 = e_i | Y_0 = e_j) = p_{ij} \ , \qquad (2.18)
$$

$$
\begin{aligned}
\Pr(O_t^i = 0|Y_0 = e_j) &= \Pr(Y_t \neq e_i, \ldots, Y_1 \neq e_i | Y_0 = e_j) \\
&= 1 - \sum_{u=1}^{t} f_{ij}(u) \ (t = 1, \ldots, T) \ , \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
\Pr(O_t^i = 1|Y_0 = e_j) &= \sum_{u=1}^{t-1} \Pr(Y_u = e_i, Y_{u-1} \neq e_i, \ldots, Y_1 \neq e_i | Y_0 = e_j) \\
&\quad \times \Pr\left(\sum_{s=u+1}^{t} \langle Y_s, e_i \rangle = 0 \middle| Y_0 = e_j \right) \\
&\quad + \Pr(Y_t = e_i, Y_{t-1} \neq e_i, \ldots, Y_1 \neq e_i | Y_0 = e_j) \\
&= \sum_{u=1}^{t-1} f_{ij}(u) \Pr(O_{t-u}^i | Y_0 = e_j) + f_{ij}(t) \ (t = 2, \ldots, T) \ , \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
\Pr(O_t^i = \tau_i|Y_0 = e_j) &= \sum_{u=1}^{t-\tau_i+1} \Pr(Y_u = e_i, Y_{u-1} \neq e_i, \ldots, Y_1 \neq e_i | Y_0 = e_j) \\
&\quad \times \Pr\left(\sum_{s=u+1}^{t} \langle Y_s, e_i \rangle = \tau_i - 1 | Y_0 = e_j \right) \\
&= \sum_{u=1}^{t-\tau_i+1} f_{ij}(u) \Pr\left(O_{t-u}^i = \tau_i - 1 | Y_0 = e_j \right) \\
&\qquad (t = 2, \ldots, T; \ \tau_i = 2, \ldots, t) \ . \qquad (2.21)
\end{aligned}
$$

Here we use the assumption that the transition probability is time-homogeneous in deriving Eqs. (2.20) and (2.21).

3 Empirical Analysis

3.1 Data

Data used in the analyses is the TOPIX daily 740 log-returns which start from January 2000 and end in December 2002. Table 1 reports the summary statistics. Since the skewness is negative and the kurtosis is higher than three, it can be said that the TOPIX log-returns in the sample period are not drawn from the unique normal distribution. Thus it would be challenging to apply HMM.

We adopt the uncollateralized overnight call rate as a risk-free asset. In the sample period, it seems to take the same value in historically very low level. Hence, for the empirical analyses purpose, the risk-free asset is assumed to be constant regardless of the

	Average (%)	Standard Deviation (%)	Skewness	Kurtosis
TOPIX	-24.12	22.65	-0.02372	4.23681

Table 1: Summary statistics for the TOPIX daily log-returns from Jan. 2000 to Dec. 2002. Here the average and the standard deviations are shown in annual rate by presuming a year has 250 business days.

state in HMM. Namely, we take 5.57×10^{-4} which is the average in the sample period from January 2000 to December 2002.

3.2 Two State HMM

In this subsection, we report the result of the empirical analysis when two state HMM is applied. Tables 2 and 3 show the estimation results for the two state HMM. From the estimated transition probability, the probability that it continues to stay in each state is high. While the probability that it transits to another state is low. These show that HMM captures the persistence in the level of volatility in the TOPIX daily log-returns.

In state 1, the asset has higher risk with higher return and the duration for this state is short. While in state 2, the asset has lower risk with lower return and the duration for this state is long. Thus, it can be said that HMM is able to characterize the asymmetric fluctuation in volatility of the TOPIX daily log-return

Also, we can visualize the above results. Figure 1 shows the time-series of the original TOPIX daily log-returns and the smoothed occupation probabilities for each state. From this figure, the occupation probability by state 2 is higher than that by state 1, in most periods.

Using these estimated parameters, we compute the European call option price with two state HMM and, compare that with the Black-Scholes model. These are reported in Table 4. In the table, the option price is the one when given the initial state, which is computed according to Eq. (2.11). Here we computed the option price in several cases when the initial price of the underlying asset is $S_0 = 100$, the time to maturity is $T = 30$, 60, 90 days, the ratio of the initial price of the underlying asset to the strike price is $S_0/K = 0.9$, 1.0, 1.1, respectively. It can be seen that the European call option price takes quite different values according to the initial state.

Figures 2 and 3 show the implied volatility curve when equating the Black-Scholes model with the option price computed from two state HMM, with the initial state being the first and second, respectively. One can see the so-called volatility smile phenomena very clearly.

3.3 Three State HMM

In this subsection, we report the result of the empirical analysis when three state HMM is applied. Tables 5 and 6 show the estimation results for three state HMM. Figure 4 shows the time-series of the original TOPIX daily log-returns and the smoothed occupation probabilities for each state. Compared to two state HMM shown in Figure 1, the expected duration for each state is very short. It can be said that the shape of the occupation probability for the first state in two state HMM is very similar to that for the first state in three state HMM. While, it will not continue to stay in the second and third states in three state HMM.

As in the analysis with two state HMM, by using the estimated parameters, we compute the European call option price with three state HMM and, compare that with the Black-Scholes model. Table 7 reports the option prices with three state HMM when given the initial state, which are computed according to Eq. (2.11). Figures 5, 6 and 7 show the implied volatility curve when equating the Black-Scholes model with the option price computed from two state HMM, with the initial state being the first, second, and third, respectively. It can be seen that the initial state in three state HMM affects more to the option price and implied volatility than that in two state HMM.

4 Conclusion and the Direction of Future Research

In this paper, we derive an analytic formula for pricing European call options under the setting of N-state Hidden Markov Models (HMM) in discrete-time framework.

When compared to the existing option pricing models which characterize stochastic volatility in asset prices, the advantages of the formula are: (1) it is an analytic formula, (2) easy to interpret its meanings and, (3) able to capture the persistence of volatility in the risky asset prices.

On estimating model parameters, we introduce the Baum-Welch algorithm with scaling in computing forward-backward probabilities to avoid underflow in computation.

We also implement empirical analyses to show the option price with HMM represents so-called volatility smile.

The following issues are left for future research. We estimate HMM when given the number of states. It would be better, however, to simultaneously estimate the optimal number of states with other parameters as in Brants (1996). Although we focus on pricing European call option, it would be very important to price more complicated option, such as Bermudan type, under the setting of HMM.

		TOPIX	
State 1	Expected Return (%)	-2.320	(3.11×10^{-3})
	Standard Deviation (%)	37.078	(9.06×10^{-5})
	Expected Duration (days)	8.5774	
State 2	Expected Return (%)	-26.580	(5.30×10^{-4})
	Standard Deviation (%)	20.368	(9.98×10^{-6})
	Expected Duration (days)	85.639	
	Log-Likelihood	2108.485	
	AIC	-4202.97	

Table 2: Estimated parameters in the two state HMM with the standard deviations in the parentheses. Data used is daily TOPIX log-returns from Jan. 2000 to Dec. 2002. Note that estimated expected log-returns and standard deviations are reported in annual rate.
The expected duration for each state is calculated as $1/(1 - p_{ii})$, where p_{ii} shows the probability of staying at the state i.

	To State 1		To State 2	
From State 1	0.883	(4.85×10^{-2})	0.117	(6.14×10^{-2})
From State 2	0.012	(6.25×10^{-3})	0.988	(3.67×10^{-2})

Table 3: Estimated transition probabilities in the two state HMM with the standard deviations in the parentheses. Data used is daily TOPIX log-returns from Jan. 2000 to Dec. 2002.

	T=30			T=60			T=90		
S_0/K	0.9	1.0	1.1	0.9	1.0	1.1	0.9	1.0	1.1
BSCall	0.3440	3.1326	9.5032	1.0731	4.4309	10.2443	1.8003	5.4269	10.9491
Call($Y_0 = e_1$)	0.6044	3.5994	9.7671	1.3221	4.7610	10.4833	2.0179	5.6852	11.1538
Call($Y_0 = e_2$)	0.146	3.0176	9.4681	1.0155	4.3213	10.1850	1.7275	5.3188	10.8778

Table 4: Numerical comparison of European call option prices between the Black-Scholes model in the first row and, the two state HMM with two different initial states in the second and third row, respectively. Data used is from Jan. 2000 to Dec. 2002.

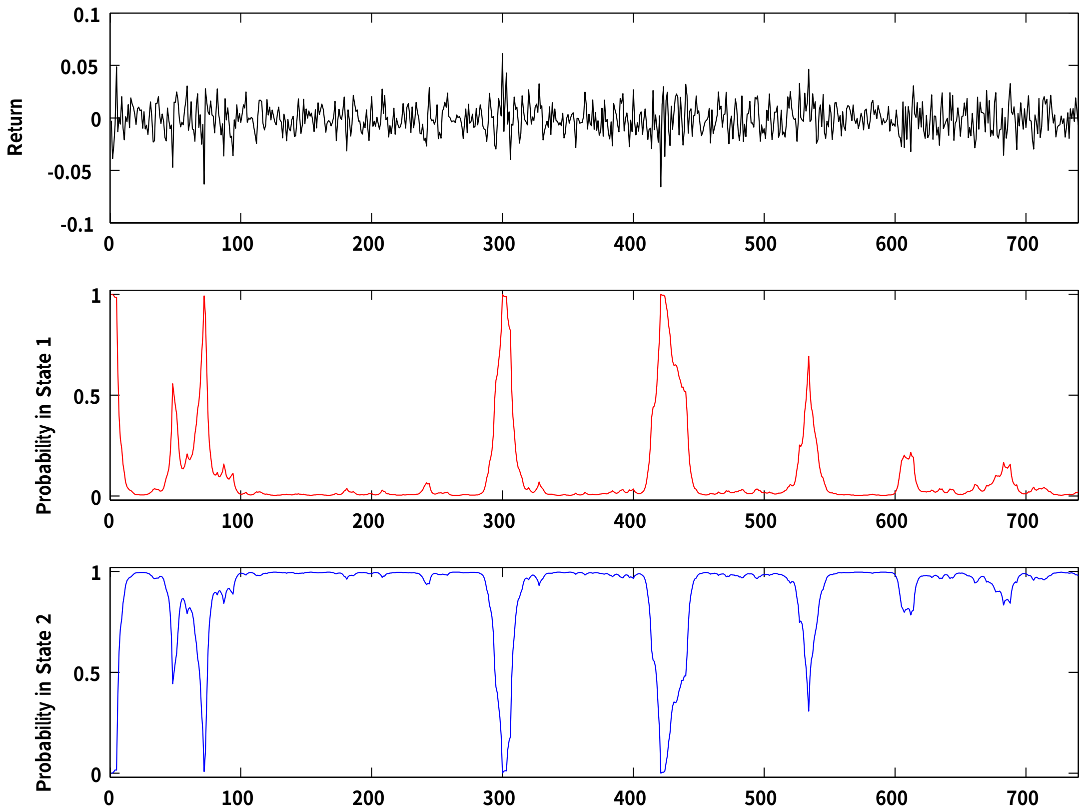

Figure 1: Estimation results for the two state HMM. The figure in the first row shows the TOPIX daily log-returns. The figures in the second and third row show the occupation probabilities by the first and second states, respectively.

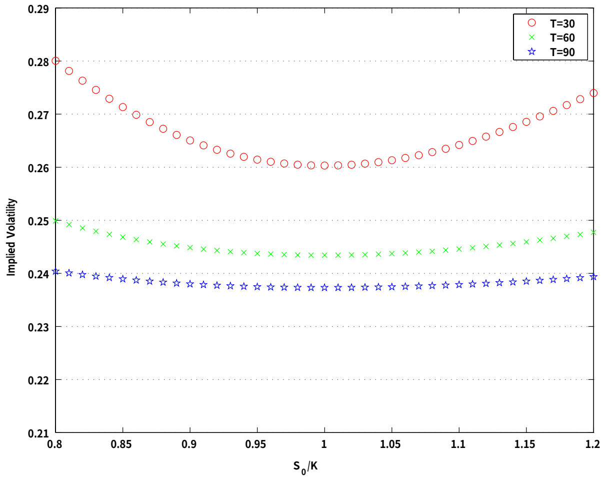

Figure 2: The figure shows the implied volatility plotted against S_0/K computed from the two state HMM with the initial state being in the first. Each curve shows the result with the maturity of 30 days (o-marked), 60 days (×-marked), and 90 days (⋆-marked), respectively.

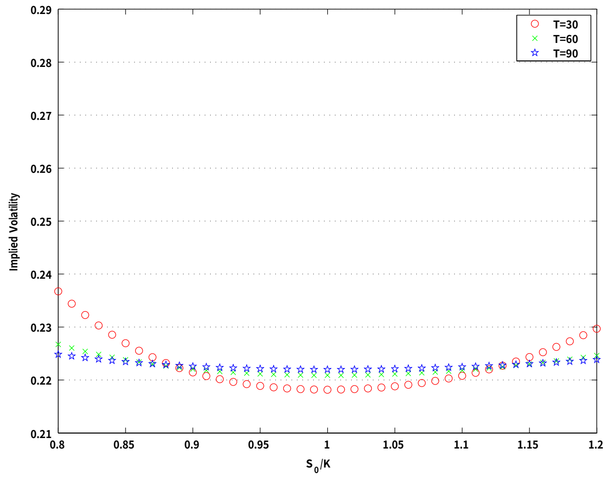

Figure 3: The figure shows the implied volatility plotted against S_0/K computed from the two state HMM with the initial state being in the second. Each curve shows the result with the maturity of 30 days (o-marked), 60 days (×-marked), and 90 days (⋆-marked), respectively.

		TOPIX	
State 1	Expected Return (%)	-43.430	(2.78×10^{-3})
	Standard Deviation (%)	34.75	(6.90×10^{-5})
	Expected Duration (days)	7.696	
State 2	Expected Return (%)	232.348	(1.20×10^{-3})
	Standard Deviation (%)	15.86	(2.08×10^{-5})
	Expected Duration (days)	2.050	
State 3	Expected Return (%)	-168.473	(7.76×10^{-4})
	Standard Deviation (%)	15.62	(1.48×10^{-5})
	Expected Duration (days)	3.538	
	Log-Likelihood	2115.120	
	AIC	-4202.24	

Table 5: Estimated parameters in the three state HMM with the standard deviations in the parentheses. Data used is daily TOPIX log-returns from Jan. 2000 to Dec. 2002. Note that estimated expected log-returns and standard deviations are reported in annual rate.
The expected duration for each state is calculated as in the Table 2.

	To State 1		To State 2		To State 3	
From State 1	0.870	(6.41×10^{-2})	0.129	(9.71×10^{-2})	0.001	(1.12×10^{-1})
From State 2	0.001	(4.02×10^{-3})	0.512	(5.77×10^{-2})	0.487	(7.06×10^{-2})
From State 3	0.032	(1.94×10^{-2})	0.251	(4.41×10^{-2})	0.717	(4.57×10^{-2})

Table 6: Estimated transition probabilities in the three state HMM with the standard deviations in the parentheses. Data used is daily TOPIX log-returns from Jan. 2000 to Dec. 2002.

	T=30			T=60			T=90		
S_0/K	0.9	1.0	1.1	0.9	1.0	1.1	0.9	1.0	1.1
BSCall	0.3440	3.1326	9.5032	1.0731	4.4309	10.2443	1.8003	5.4269	10.9491
Call($Y_0 = e_1$)	0.3856	3.1073	9.5377	0.8987	4.1059	10.0663	1.4317	4.9054	10.5916
Call($Y_0 = e_2$)	0.0676	2.2256	9.1923	0.3528	3.1528	9.5171	0.7160	3.8643	9.8944
Call($Y_0 = e_3$)	0.0645	2.2115	9.1886	0.3384	3.1208	9.5019	0.6900	3.8203	9.8684

Table 7: Numerical comparison of European call option prices between the Black-Scholes model in the first row and, the three state HMM with three different initial states in the second, third, and fourth row, respectively. Data used is from Jan. 2000 to Dec. 2002.

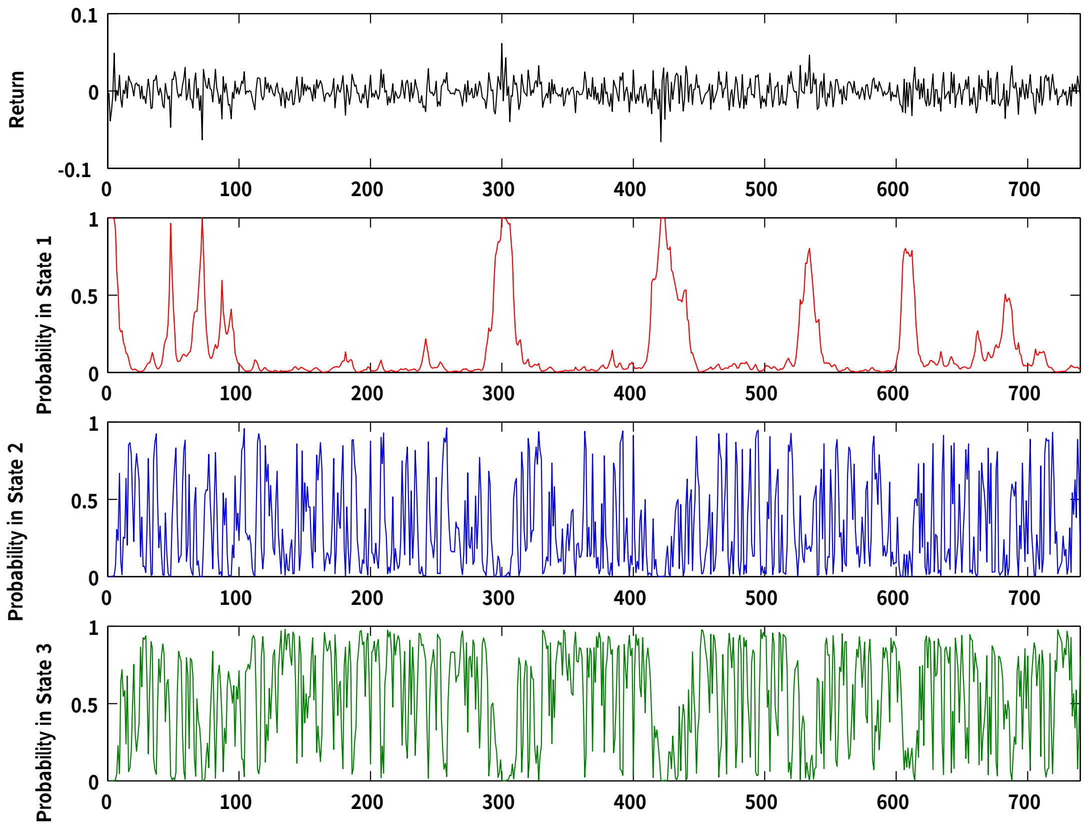

Figure 4: Estimation results for the three state HMM. The figure in the first row shows the TOPIX daily log-returns. The figures in the second, third, and fourth row show the occupation probabilities by the first, second, and third states, respectively.

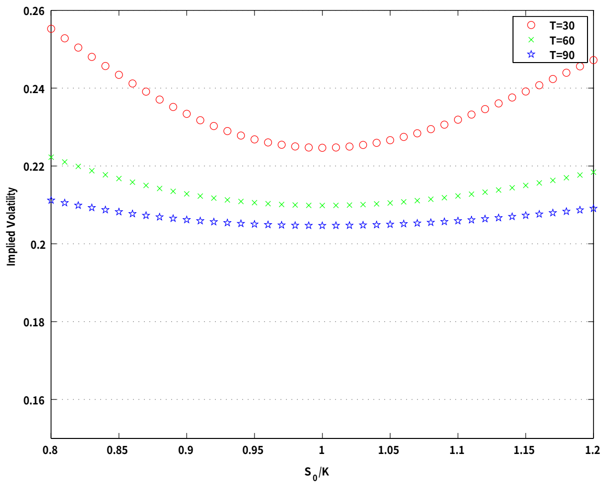

Figure 5: The figure shows the implied volatility plotted against S_0/K computed from the three state HMM with the initial state being in the first. Each curve shows the result with the maturity of 30 days (o-marked), 60 days ($\times$-marked), and 90 days ($\star$-marked), respectively.

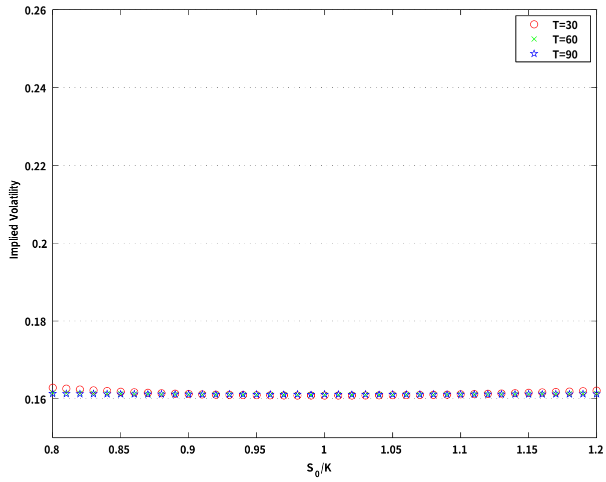

Figure 6: The figure shows the implied volatility plotted against S_0/K computed from the three state HMM with the initial state being in the second. Each curve shows the result with the maturity of 30 days (o-marked), 60 days ($\times$-marked), and 90 days ($\star$-marked), respectively.

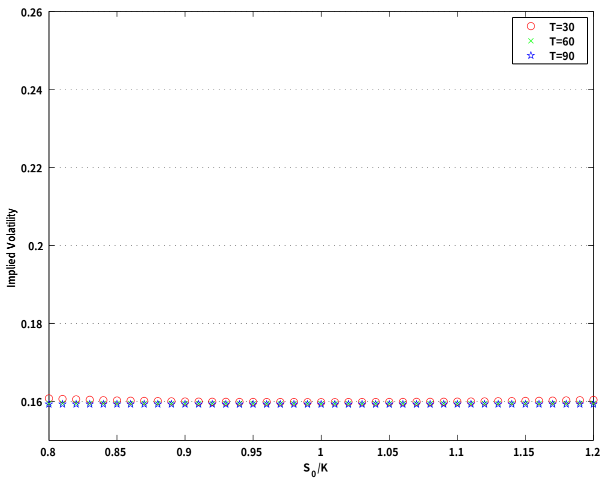

Figure 7: The figure shows the implied volatility plotted against S_0/K computed from the three state HMM with the initial state being in the third. Each curve shows the result with the maturity of 30 days (o-marked), 60 days ($\times$-marked), and 90 days ($\star$-marked), respectively.

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