Price Manipulation with Positive Feedback Traders*

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ABSTRACT

This paper investigates the price manipulation that exploits *positive feedback traders* in a continuous auction market. I characterize the class of pricing rules of competitive liquidity providers that prevent the price manipulation, where the pricing rule is a function that maps an aggregate market order to a uniform trading price. The main result is to show that a broad and plausible class of pricing rules cannot prevent price manipulation by a large speculator.

Key Words: market microstructure, price manipulation, positive feedback trader, price impact.
Journal of Economic Literature Classification Numbers: G11, D49.

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1 Introduction

Price manipulation is a speculative strategy that artificially causes differences over time between purchase and sale prices and provides speculators with a positive profit. This paper introduces a new strategy for price manipulation: the price manipulation that exploits momentum trading by frenzied traders. I consider such price manipulation by a simple model that describes a single stock trading in a *market making system*.

The market in my model is composed of N discrete trading periods. There are three kinds of market participants: a large speculator, competitive liquidity providers, and *positive feedback traders* who act as frenzied momentum traders (e.g., De Long et al. (1990)). The trading rule in the market is a continuous auction system. In each period, the speculator and positive feedback traders place a market order simultaneously, then liquidity providers simultaneously offer a price for the aggregate order and execute it by a competitively determined single price.

A price offered by a liquidity provider is determined by his/her *pricing rule*, which is a function that maps an aggregate order to an offered price. The model assumes that the liquidity provider sets a pricing rule before the beginning of the initial period of the market and commits to use it until the end of N period trading.

The behavior of positive feedback traders is determined by a price trend. The price gain of the stock in the previous period leads to them purchasing the stock in the present period and their purchase of the stock implies a further price gain in the next period. If there is no exogenous shock, this self-feeding behavior raises the market price continuously and its price path may behave as a bubble.[1]

[1] Shiller (2008) states the following regarding the price increases observed in the 2007–08 subprime loan tragedy:

> An important part of what happens during a speculative bubble is mediated ... by the prices that are observed there and subsequently amplified by the news media. ... The media weave stories around price movements, and when those movements are upward, the media tend to embellish and legitimize "new era" stories with extra attention and detail. Feedback loops appear, as price increases encourage belief in "new era" stories,... and so lead to further

This paper analyzes whether a speculator can earn a positive profit through speculation in terms of price manipulation that exploits the self-feeding behavior of positive feedback traders. The speculator may earn a positive profit by a "simple" strategy such that he/she initially buys/sells a large amount of a stock to deliberately increase/decrease the market price of the stock and then sells/buys the shares back at the time that the price becomes sufficiently high/low owing to positive feedback traders.[2]

This paper characterizes the set of *linear* pricing rules that prevent such price manipulation.[3] I focus on four types of price manipulation using positive feedback traders.

The first and main analysis considers a one time buy and sell-back strategy, which I call a *simple strategy*, under time-independent pricing rule. A pricing rule is *time-independent* if the same market order corresponds to the same price in any period. If a pricing rule prevents such price manipulating simple strategies in any period market, then we call the rule an *α-manipulation-proof* pricing rule. An α-manipulation-proof pricing rule enables the speculator to earn at best a non-positive payoff from simple strategies regardless of the number of trading periods. In this paper, I formulate a necessary and a sufficient condition for α-manipulation-proof pricing rules (Theorem 1).

Second, I characterize a sufficient condition for time-independent manipulation-proof pricing rules against *any* speculative strategy that begins and ends with zero position in a fixed N-period market, which we call such a pricing rule a *β-manipulation-proof* pricing rule (Theorem 2).

Third, I consider a price manipulating strategy that is composed of a single purchase and sale in two consecutive periods in a fixed N-period

> price increases. The price-story-price loop repeats again and again during a speculative bubble. (pp. 45–46).

[2]Note that the decision for this speculation is independent of the fundamental value of the trading stock. Judging from the trading strategy described in Soros (2003, pp. 49–72), his speculative strategy in stock markets exploits trend-chasing traders. In fact, his strategy has earned him large profits and prices were bubbling. Importantly, his strategy does not rely on fundamentals, but rather on the anticipation of the trend-chasing behavior of traders.

[3]The reason why we focus on linear pricing rules is described in the next section.

market under *time-dependent* pricing rules. If a time-dependent and linear pricing rule prevents such strategies in the market, then we call the rule a *γ-manipulation-proof* pricing rule. I formulate a necessary and sufficient condition for γ-manipulation-proof pricing rules (Theorem 3). In addition, I apply the results to the trading model of Kyle (1985) and investigate whether liquidity providers follow a γ-manipulation-proof pricing strategy in equilibrium. I point out that competitive pricing of liquidity providers may not be sufficient to prevent price manipulation. To prevent price manipulation, liquidity providers must make sufficiently correct estimation of the trading volume from positive feedback traders; if they do not aware of positive feedback traders at all, the speculator may undertake price manipulation (Propositions 7 and 8).

Last, I consider a risk on price manipulation by introducing a stochastic noise trader, which is independently defined of positive feedback traders but affects their trading volume through prices. I evaluate the performance of price manipulation by the *Sharpe ratio*. If a time-independent and linear pricing rule makes the Sharpe ratio obtained from a simple strategy be bounded when the trade size of the speculator goes infinity, then we call the rule a *δ-manipulation-proof* pricing rule. I show that the γ-manipulation-proof pricing rule is equivalent to the α-manipulation-proof pricing rule (Theorem 4).

1.1 Related literature

Price manipulation in stock markets has been investigated in several settings. Jarrow (1992) investigates the possibility of price manipulation by a large uninformed trader in a general frictionless market. Chakraborty and Yilmaz (2004b) investigate the price manipulation that exploits asymmetric information in a discrete version of the Kyle (1985) model.[4] Huberman and Stanzl (2004) consider price manipulation in a continuous auction market as in Kyle (1985) and examine the relation between pricing rules of liquidity

[4]Chakraborty and Yilmaz (2004a) investigates the price manipulation in the Glosten–Milgrom model (Glosten and Milgrom (1985)).

providers and the possibility of price manipulation. They show that no price manipulation can be achieved if a pricing rule is *linear* when (i) there is no noise trader, or (ii) the trading volume of noise traders follows normal distribution. My model is based on Huberman and Stanzl (2004) for comparison the obtained results and I show that most linear pricing rules *cannot* prevent price manipulation if (iii) there are positive feedback traders. This is the central result of my paper. As I shall explain, this result is due to the fact that positive feedback traders make a pricing rule *nonlinear* from the speculator's point of view.

The positive feedback trader is first modeled by De Long et al (1990).[5] They consider a model without middlemen and show that positive feedback traders are the source of price bubbles. When the speculator expects the fundamental value to be high, the speculator's optimal decision triggers a positive feedback trader's purchase in equilibrium, which makes the market price surpass the fundamental value of the trading stock. On the other hand, I states that mispricing can emerge regardless of fundamental values under some pricing rules. At the same time, however, my result suggests that liquidity providers adjust prices if they are aware of positive feedback traders. Hence the existence of positive feedback traders does not always imply the emergence of mispricing.

2 Model

2.1 Preliminaries

Consider a single stock market with N discrete finite trading periods $n = 1, 2, \cdots, N$ in the time interval $(0, 1] \subset \mathbb{R}_+$. The stock can be bought or sold via market orders in each period n defined by n/N, where $1 \leq n \leq N$.

There are three kinds of market participants: a speculator, competitive risk-neutral liquidity providers, and positive feedback traders. In each trading period n, the speculator and positive feedback traders simultaneously

[5] Hong and Stein (1999) and Barberis and Shleifer (2003) explain the emergence of positive feedback trading.

place a market order x_n and ξ_n, respectively, where $x_n, \xi_n \in \mathbb{R}$. The liquidity provider offers a price $p_n \in \mathbb{R}_+$ for the placed aggregate order $q_n \equiv x_n + \xi_n$ according to his/her *pricing rule*. We focus on the pricing rules that follow the form

$$\begin{aligned} p_n &= p_{n-1} + P_n(q_n) - P_{n-1}(q_{n-1}) + U_{n-1}(q_{n-1}) \\ &= p_0 + \sum_{k=1}^{n-1} U_k(q_k) + P_n(q_n), \end{aligned} \tag{1}$$

where $p_0 \in \mathbb{R}_+$ is an opening price in the market. This price form follows Huberman and Stanzl (2004).[6] The *price impact function* $P_n(q_n)$ captures the immediate price reaction to the market order q_n. The *price update function* $U_n(q_n)$ captures only the permanent price impact from q_n. Liquidity providers simultaneously choose $\{U_k, P_k\}_{k=1}^{N}$ in period 0.

The decision of positive feedback traders depends on the latest price difference (trend): The aggregate order of positive feedback traders follows $\xi_1 = 0$ and

$$\xi_n = \beta(p_{n-1} - p_{n-2}) \tag{2}$$

with a momentum magnitude $\beta > 0$. The behavior defined by Eq.(2) is the same as in De Long et al. (1990).

I consider the situation in which the speculator correctly estimates $\{U_k, P_k\}$ and β in period 0, i.e., he can anticipate the pricing rule of liquidity providers and the trading volume of positive feedback traders. Let $\boldsymbol{x} = (x_1, \cdots, x_N)$ denote a *strategy* of the speculator. $x_n > 0$ (< 0) denotes a buy (sell) order in period n. The payoff of the speculator is

$$\pi(\boldsymbol{x}; N) \equiv -\sum_{n=1}^{N} p_n x_n.$$

[6]Huberman and Stanzl (2004) divide a market price into two parts: the quote price $\tilde{p}_n = \tilde{p}_{n-1} + U_{n-1}(q_{n-1}) + \epsilon_n$, and the trading price $p_n = \tilde{p}_n + P_n(q_n)$, where ϵ_n is a noise term, which is omitted in my model.

We impose the constraint

$$\sum_{n=1}^{N} x_n = 0 \tag{3}$$

for any strategy $\boldsymbol{x}$. Eq.(3) means that the speculator closes out a trading with zero position. In accordance with Huberman and Stanzl (2004), we call a strategy satisfying Eq.(3) a *round-trip* strategy. A (risk-neutral) *price manipulation* is a round-trip strategy that makes $\pi(\boldsymbol{x}; N) > 0$ for some N.

2.2 Intuition of the main result

Using a simple example, I provide intuition as to why positive feedback traders generate a speculative opportunity for the speculator. When there are no positive feedback traders and any other noise traders, Huberman and Stanzl (2004, Proposition 1) show that only the linear pricing rules can prevent price manipulation with round-trip strategies. Hence a nonlinear pricing rule induces the speculator to implement price manipulation. The following example points out that positive feedback traders cause a *nonlinear effect* to linear pricing rules from the speculator's point of view.

As a benchmark, we first consider the case of no positive feedback traders. For simplicity, we assume the following situation.

- The participants are a single speculator and a single liquidity provider.
- $N = 8$.
- The liquidity provider follows the price functions in Eq.(1) such that $U(q_n) = aq_n$ if $q_n \geq 0$ and $U(q_n) = bq_n$ if $q_n < 0$ with $a > b > 0$ and $U = P$.

Clearly this price function U is not linear. Then the speculator can earn a positive profit for some (a, b) by implementing a round-trip strategy.

Consider the following round-trip strategy: $x_n = 1$ for $n = 1, 2, 3, 4$ and $x_n = -1$ for $n = 5, 6, 7, 8$. Figure 1 exhibits a typical price path that brings the speculator a positive profit by the round-trip strategy. A bold slope line with circles indicates the price path with this nonlinear pricing rule. Each circle indicates the price in each trading period. The dashed line with boxes indicates the price path with the price function $U(q_n) = aq_n$ for $n = 5, 6, 7, 8$. Figure 1 indicates that $l_1 = |p_5 - p_4|$, $l_2 = |p_7 - p_1|$, $p_8 = p_2$, and $p_6 = p_3$. Hence the speculator earns the positive payoff represented by $l_2 - l_1$, which implies a loss for the liquidity provider. Figure 1 also indicates that, for the symmetric linear function $U(q_n) = aq_n$ for $n = 5, 6, 7, 8$, the speculator cannot gain from the same round-trip strategy.

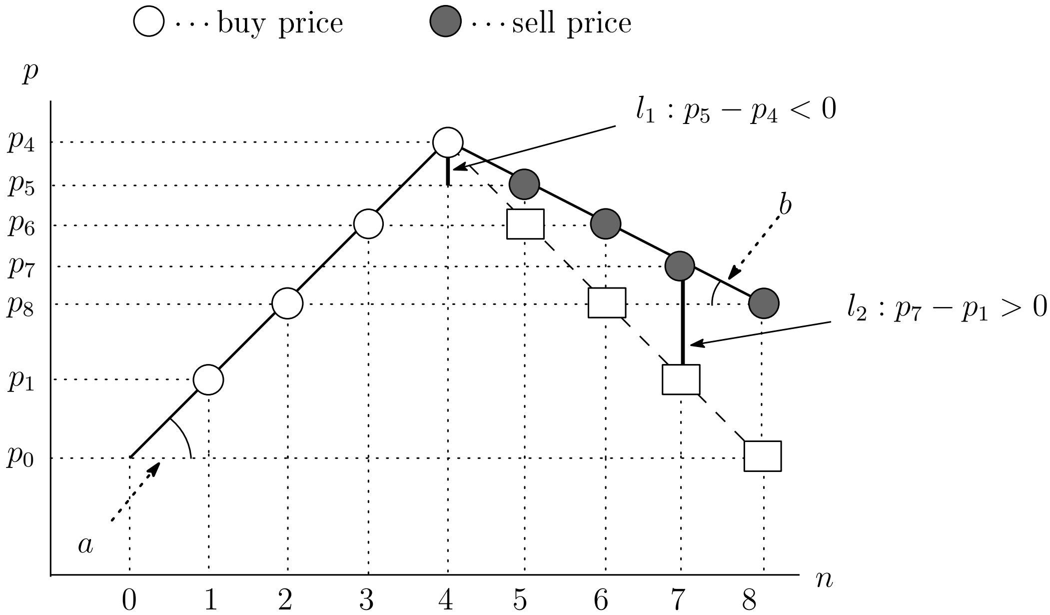

Figure 1: A nonlinear pricing strategy may motivate the speculator to implement price manipulation. The white circles indicate buy prices; the black circles sell prices. The square boxes indicate sell prices for the linear pricing rule with $U(q) = aq$ for all $q \in \mathbb{R}$. The figure shows the speculator gains from the round-trip strategy because l_2 is greater than l_1.

Next, we introduce the positive feedback traders defined by Eq.(2) in the following eight-period model.

- $U_n(q_n) = P_n(q_n) = \lambda q_n$ for all $n \in \{1, \cdots, 8\}$, where $\lambda \in \{11/30, 1/4, 1/8\}$.
- $\beta = 3$, $p_0 = 1$.

Suppose that the speculator implements the round-trip strategy $\boldsymbol{x}'$ such that $x'_1 = 1$, $x'_8 = -1$, and $x'_n = 0$ for all other n. It generates the demand function of positive feedback traders $\xi_n = (\beta\lambda)^{n-1} = (3\lambda)^{n-1}$ for $n \geq 2$. Figures 2 to 4 depict the price paths when λ takes the values $11/30, 1/4, 1/8$, respectively. Figures 2 and 3 show that the speculator earns a positive profit through strategy $\boldsymbol{x}'$ when $\lambda = 11/30$ and $1/4$ (sufficiently large), which is impossible when there is no positive feedback trader. Figure 4, however, shows that the speculator cannot gain from strategy $\boldsymbol{x}'$ if $\lambda = 1/8$ (sufficiently small).

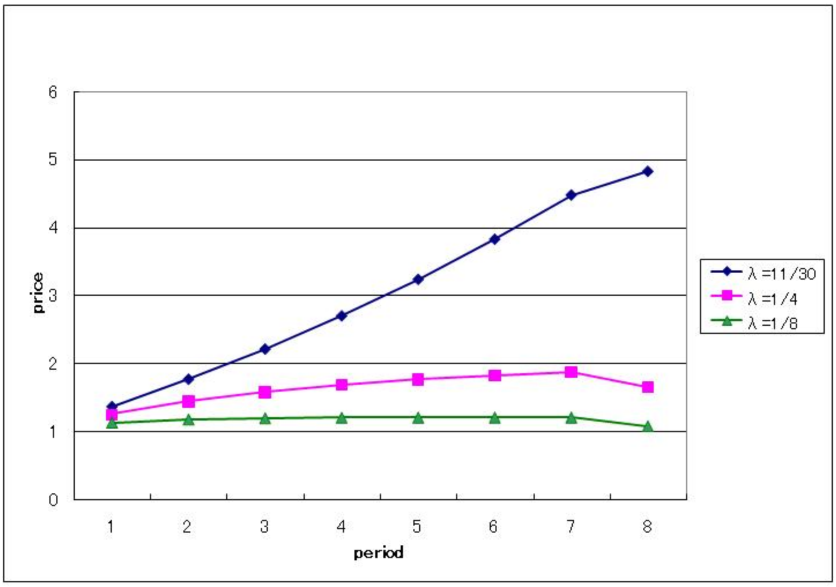

Figure 2: Top: $\lambda = 11/30$, middle: $\lambda = 1/4$, bottom: $\lambda = 1/8$. $p_0 = 1$ and $\beta = 3$. The speculator gets a payoff of approximately 3.46 and 0.4 when $\lambda = 11/30$ and $\lambda = 1/4$ respectively by the strategy $x_1' = 1$. When $\lambda = 1/8$, the payoff is approximately -0.05.

These figures, together with the previous example without positive feedback traders, suggest that positive feedback traders generate a *nonlinear* effect on linear pricing rules from the speculator's point of view. Indeed the price impact is λ when the speculator buys 1 unit of the stock but $\lambda(-1)+\lambda\xi_8$ *for the speculator* when he/she sells 1.

We divide such a nonlinear effect into two effects: the cancel effect and the rising effect. Suppose, for example, that the speculator implements the strategy such that $x_1 = 1$ and $x_2 = -1$. A buy order from positive feedback traders in period 2 is expected to raise the trading price in period 2 by $\mu\xi_2$. The *cancel effect* means the buy order $\mu\xi_2$ helps to cancel out the price drop

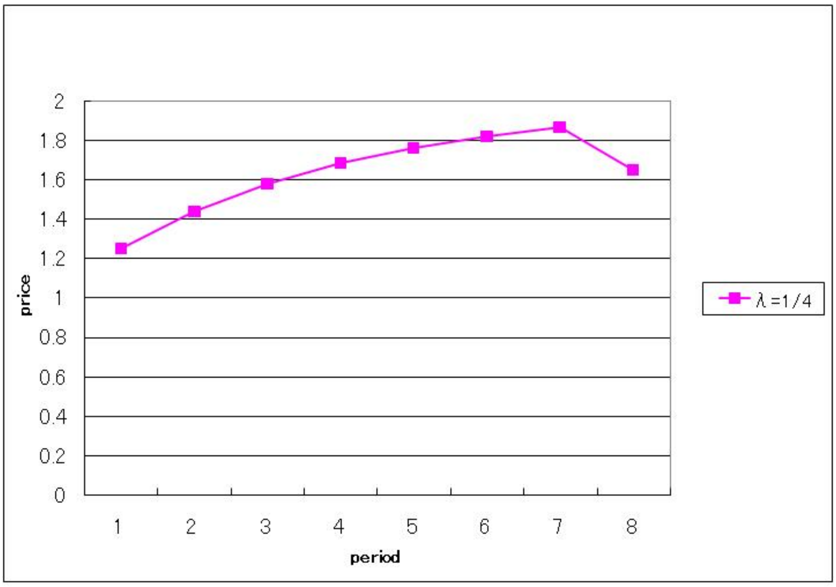

Figure 3: $\lambda = 1/4$. The speculator gets a positive payoff because p_8 is greater than p_1.

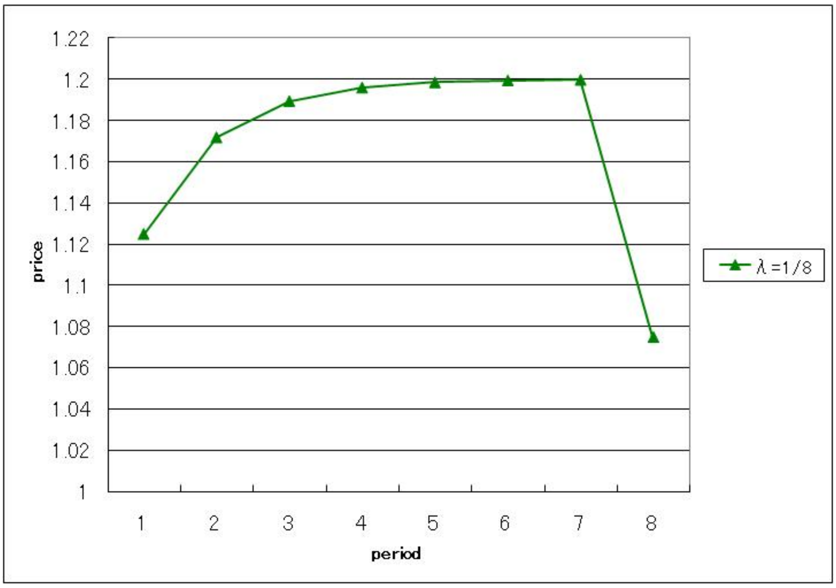

Figure 4: $\lambda = 1/8$. The speculator gets a negative payoff from the strategy $x'_1 = 1$ because $p_8 < p_1$.

$\mu(-1)$ from the speculator's sale -1 in period 2. Thus, the speculator can earn a positive profit if the cancel effect is sufficiently strong (see Figure 5).

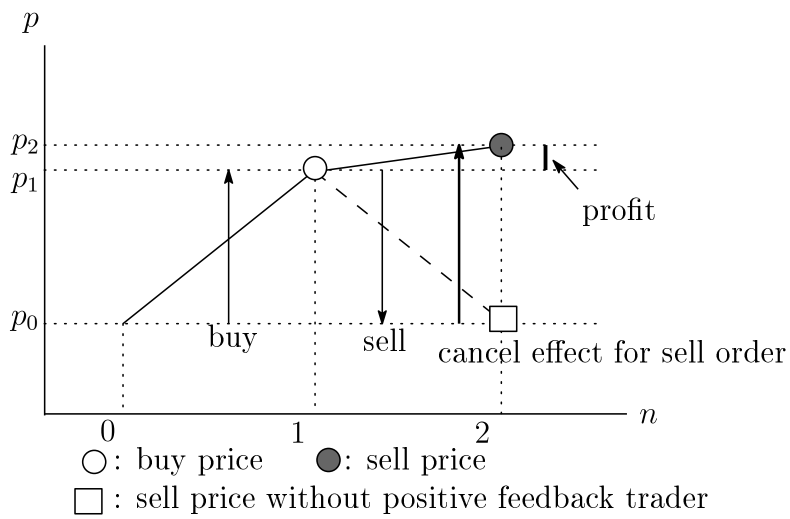

Figure 5: A strong cancel effect brings a positive profit for the speculator.

The *rising effect* is simpler. Back to our example and consider strategy $\boldsymbol{x}'$. Then, positive feedback traders raise prices monotonically by their self-feeding behavior until period 8. As we saw in Figure 3, the rising effect may create a gain even if the cancel effect is too weak to cancel out the price drop in selling the stock.

These nonlinear effects are major sources of speculative opportunities. In this paper, we mainly consider the most basic form of pricing rule—*time-independent* and *linear* price functions—and characterize the set of price functions that make both the effects be negligible as in Figure 4. In particular, we seek the price function that prevents price manipulation *for all* N. When $N = 2$, only the cancel effect matters. As N increases, we can see the cancel effect be weaker and the rising effect be stronger in our example $U_n = P_n$. To purely investigates the relation between these effects and N, we mainly consider a "simple" class of strategies such that $x_1 = x \in \mathbb{R}$, $x_N = -x$, and $x_n = 0$ for all other n.

3 Main Results

3.1 Simple strategies and time-independent and linear price functions

Assumption 1 *We assume the functions $U_n(q_n)$ and $P_n(q_n)$ to be* time-independent *and* linear*, i.e.,*

$$U_n(q_n) = \lambda q_n, \qquad P_n(q_n) = \mu q_n,$$

where $(\lambda, \mu) \in \mathbb{R}_+^2$

Under Assumption 1, we call (λ, μ) a pair of *price coefficients.* A pricing rule is said to be *linear* when both $U_n(q_n)$ and $P_n(q_n)$ are linear for all n.

A round-trip strategy $\boldsymbol{x} = (x_1, \cdots, x_N)$ is said to be a *simple strategy* when $x_1 = x \in \mathbb{R}$, $x_N = -x$, and $x_n = 0$ for all other n. The main analysis in this section is to characterize the pair of price coefficients (λ, μ) that prevents price manipulation with simple strategies for all N.

Definition 1 *A pair of price coefficients (λ, μ) has the* α-manipulation-proof property *if, for all $x \in \mathbb{R}$, we obtain*

$$\sup_{N \geq 1} (p_N - p_1)x \leq 0,$$

where p_1 and p_N are defined by Eq.(1) with Assumption 1.

The α-manipulation-proof property is the property that prevents price manipulation with simple strategies regardless of trading opportunities N. If (λ, μ) does not have the α-manipulation-proof property, then there exists N such that the speculator can earn a positive profit by implementing a simple strategy in the N-period market. Hence the α-manipulation-proof property provides us with one criterion how much we have to restrict pricing rules in order to hold down the rising effect.

We call the pair of price coefficients that have the α-manipulation-proof property *α-manipulation-proof price coefficients.* The *α-manipulation-proof set* is the set of α-manipulation-proof price coefficients.

3.1.1 Result

Theorem 1 *If (λ, μ) has the α-manipulation-proof property, then (λ, μ) satisfies the following inequalities.*

$$0 \leq \beta\mu - 1 < \beta\lambda, \quad 0 \leq \beta\lambda < 1, \quad \lambda - 2\mu + 2\beta\lambda\mu \leq 0. \tag{4}$$

Conversely, if $(\lambda, \mu) \in \mathbb{R}_+^2$ satisfies Eq.(4) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda \geq 0$, then (λ, μ) has the α-manipulation-proof property.

Proof. For notational convenience, let $D = (\beta\mu)^2 - 4\beta\mu + 4\beta\lambda$ and $K = \beta\mu$. Here I sketch a proof for the case $D > 0$. A full proof is in the Appendix.

(1) Market orders under simple strategies:
Let x denote a simple strategy. Assume that $x > 0$ (a symmetric argument holds when $x < 0$). A simple strategy x generates the positive feedback traders' aggregate demand as follows: $\xi_0 = \xi_1 = 0$, $\xi_2 = \beta\mu x$, and

$$\xi_n = \beta\mu\xi_{n-1} + \beta(\lambda - \mu)\xi_{n-2}, \tag{5}$$

for $n \in \{3, \cdots, N-1\}$. Eq.(5) represents a second-order linear homogeneous difference equation. Section A.1 in the Appendix shows that

$$q_n = \frac{x}{\sqrt{D}} \left\{ \left(\frac{K + \sqrt{D}}{2} \right)^n - \left(\frac{K - \sqrt{D}}{2} \right)^n \right\} \tag{6}$$

for all $n \in \{1, \cdots, N-1\}$ with initial values $q_0 = 0$ and $q_1 = x$. Note that $q_n = \xi_n$ when $n \in \{2, \cdots, N-1\}$. This is the equation of market orders with simple strategy x when $n \in \{1, \cdots, N-1\}$.

(2) The payoff of the speculator and the α-manipulation-proof property:
Let $f_n = \left(\frac{K+\sqrt{D}}{2}\right)^n - \left(\frac{K-\sqrt{D}}{2}\right)^n$ for notational convenience. When $D > 0$, Lemma A.1 in the Appendix shows that $f_n > 0$ for all $n \in \mathbb{Z}_+$. The payoff

function of the speculator from a simple strategy x is

$$(p_N - p_1)x = x^2 \left\{ \frac{1}{\sqrt{D}} \left(\lambda \sum_{n=1}^{N-1} f_n + \mu f_N \right) - 2\mu \right\}. \tag{7}$$

We can see that Eq.(7) increases monotonically with N when $f_n > 0$. Hence, in the case that $D > 0$, it is necessary and sufficient for (λ, μ) to have the α-manipulation-proof property that

$$\lim_{N\to\infty} (p_N - p_1)x \leq 0. \tag{8}$$

Lemma A.3 in the Appendix shows that Eq.(8) is equivalent to (λ, μ) satisfying Eq.(4). ■

In almost the same way, we can prove the theorem in the case $D = 0$. A problem arises, however, when we consider the case $D < 0$. See Figure 6. It demonstrates the case $N = 30$, $p_0 = 1$, $\beta = .5$, and $x = 1$ when (λ, μ) satisfies $D < 0$. The figure indicates that prices fluctuate with n and suggests that the payoff of the speculator does not increase monotonically with N, i.e., it is better for the speculator to sell in period $k^* < N$ in an N-period market. This is why it is difficult to characterize a necessary *and* sufficient condition for the α-manipulation-proof property. Indeed, Lemma A.6 in the Appendix proves that there are infinitely many Ns that satisfy $p_N > \lim_{n\to\infty} p_n$ when (λ, μ) satisfies $D < 0$.

3.1.2 Geometric property

Figure 7 indicates a graphical image of Theorem 1. For comparison, we consider the price coefficients that make the speculator lose by trading when $N = 2$. By easy calculation, we obtain the following result. For use in later sections, I summarize the result as a proposition.

Proposition 1 *When $N = 2$. The (risk-neutral) speculator does not imple-*

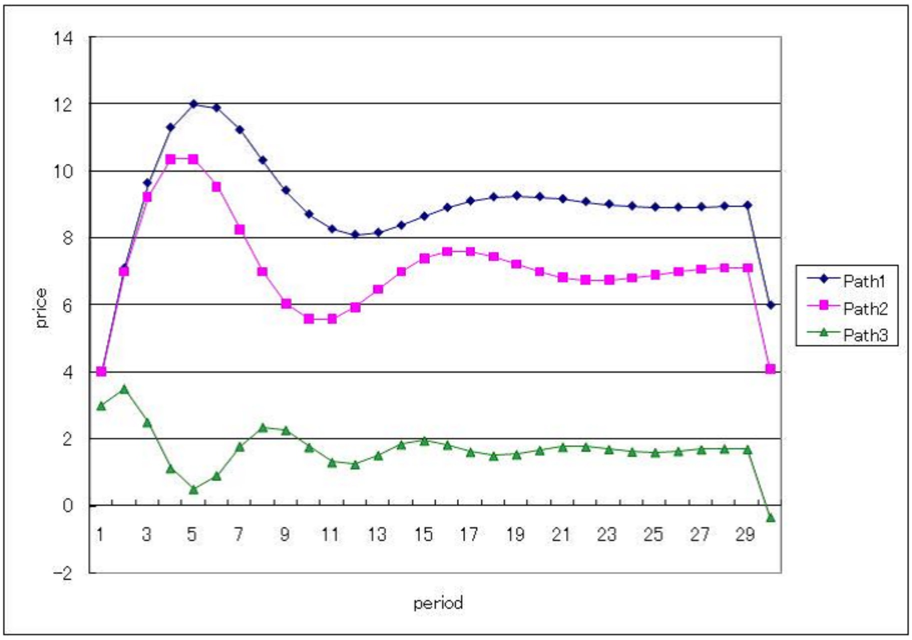

Figure 6: Common parameters:$(N, p_0, \beta) = (30, 1, 0.5)$. For (λ, μ), Path1 is $(1.6, 3)$, Path2 is $(1.5, 3)$, and Path3 is $(0.5, 2)$. The speculator implements the simple strategy $x = 1$. $(\lambda, \mu) = (1.6, 3)$, $(1.5, 3)$, $(0.5, 2)$ satisfy $\lambda - 2\mu + 2\beta\lambda\mu > 0, = 0, < 0$, respectively.

ment simple strategies if and only if

$$(\lambda, \mu) \in C \equiv \{(\lambda, \mu) \mid \beta\mu^2 - 2\mu + \lambda \leq 0\}.$$

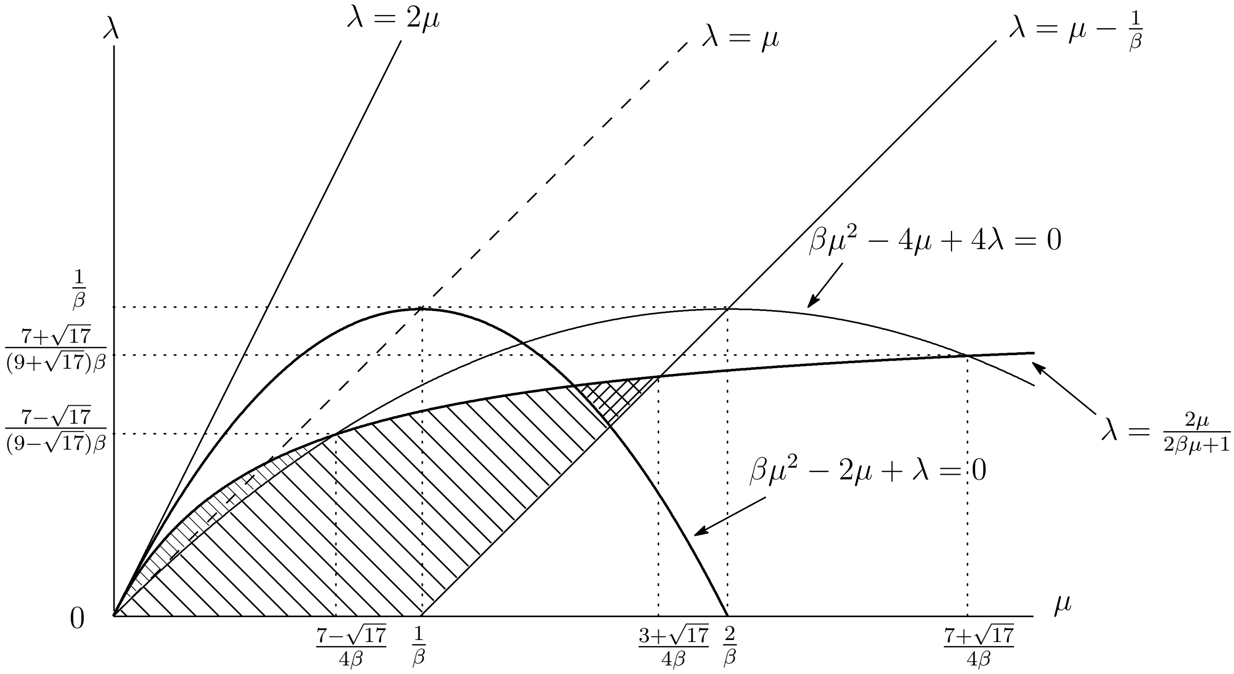

Figure 7: A pair of price coefficients (λ, μ) in the finer shaded area with a single directed line has the manipulation-proof property. A pair of manipulation-proof price coefficients must be in the union of the finer and the coarser shaded areas except for the cross lines area.

Figure 7 describes the shape of set C. As I mentioned in the previous section, the rising effect becomes stronger and the cancel effect becomes weaker when N increases. In fact, the rising effect relates mainly to the price-update coefficient λ, while the cancel effect relates mainly to the price-impact coefficient μ. This is why relatively large λ does not satisfy Eq.(4), while it is included in C.

The same reasoning can be applied to explain why the α-manipulation-proof price-update coefficient λ is relatively smaller on average than the manipulation-proof price-impact coefficient μ, as in Figure 7. We note the following inequality

$$|p_0 + \lambda \sum_{k=1}^{N-1} \xi_k + \mu \xi_N| \leq p_0 + \lambda \sum_{k=1}^{N-1} |\xi_k| + \mu |\xi_N| \equiv \hat{p}_N.$$

When $D > 0$, $\xi_n \geq 0$ for all n and hence $\hat{p}_N = p_N$. To obtain the α-manipulation-proof property, $\hat{p}_N$ should be converged. Hence ξ_N must be smaller as N becomes large, which means the cancel effect gets weaker as N gets large. On the other hand, $\sum_{k=1}^{N-1} |\xi_k|$ increases monotonically with N. These facts imply that the rising effect accounts for a large part of the nonlinear effect when N is large. Thus it is reasonable to set λ relatively smaller than μ to achieve the α-manipulation-proof property because the α-manipulation-proof property requires (λ, μ) to satisfy $(p_N - p_1)x \leq 0$ *for all* N.

Let us denote by S^* the set of α-manipulation-proof price coefficients. In Figure 7, we see that the finer and coarser shaded areas with single lines, say S_1 and S_2 respectively, and the shaded area with cross lines, say S_3. S_1 indicates the set that is defined by Eq.(4) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda \geq 0$. The union of S_1, S_2, and S_3 in the figure corresponds to the area defined by Eq.(4). We see the area S_1 is contained within the area C, but C does not include the areas S_1, S_2, and S_3. By definition, the price coefficients in S_3 cannot be α-manipulation-proof price coefficients. These facts imply that $S^* \subset S_1 \cup (S_2 \backslash S_3)$, but we do not know whether $S_1 \subsetneq S^*$ or not.

We can easily check that the α-manipulation-proof set becomes large as $\beta \to 0$. In the limit, we obtain the set

$$A = \{(\lambda, \mu) \mid \lambda \leq 2\mu, \ \lambda \geq 0, \ \mu \geq 0\}.$$

In fact, we can easily check that the speculator cannot earn a positive profit by any simple strategy in any period market when (λ, μ) is in A and $\beta = 0$.[7] In addition, we can also easily check that the speculator can earn a positive profit when $(\lambda, \mu) \in \mathbb{R}_+^2 \backslash A$ regardless of N and β. We summarize the result

[7] A result of Huberman and Stanzl (2004, Proposition 3) implies that set A ensures that the speculator cannot earn a positive profit by any round-trip strategy.

as a proposition.

Proposition 2 *The α-manipulation-proof set is defined on A.*

It is trivial that $(\lambda, \mu) = (0, 0)$ is a pair of α-manipulation proof price coefficients. We consider the question whether we can find a nontrivial α-manipulation-proof pricing rule for any $\beta > 0$? The answer is yes.

Proposition 3 *For all $\beta > 0$, there exists a pair of α-manipulation-proof price coefficients $(\lambda, \mu) > (0, 0)$.*

Proof. See Lemma A.7 in the Appendix. ■

Lemma A.7 also shows that Lebesgue measure of the α-manipulation-proof set is larger than zero.

If (λ, μ) makes price paths be unbounded for a simple strategy $x < \infty$, then the speculator is profitable by implementing simple strategy x. Hence any plausible pair of price coefficients should make price paths be bounded. The following set B is the desirable one.

Proposition 4 *If (λ, μ) has the α-manipulation-proof property, then (λ, μ) must be in B, where*

$$B = \left\{ (\lambda, \mu) \;\middle|\; 0 \leq \lambda < \frac{1}{\beta},\ 0 \leq \mu < \frac{2}{\beta},\ \lambda \leq 2\mu \right\}. \tag{9}$$

Proof. See Lemma A.8 in the Appendix. ■

Note that $B = A$ in the limit of $\beta \to 0$.

We can compare the size of the α-manipulation-proof set and of set B by its measure. The next result states that, compared with set B, the requirement of the α-manipulation-proof property restricts drastically the feasible price coefficients *even if we only take into account simple strategies.*

Proposition 5 *S_1 accounts for approximately only 2.6% of B, while S_2 accounts for approximately 41.0% of B.*

Proof. See Lemma A.9 in the Appendix. ■

Note that both the percentages are independent of β.

3.1.3 Application: speculator as an arbitrageur

Whereas the speculator does not undertake price manipulation under a manipulation-proof pricing rule, he/she may participate in trade as an arbitrageur. To see it by a simple model, we assume the following.

- $N = 3$.
- The trading stock has a liquidation value v, which is not public until the beginning of period 3.
- The speculator privately knows the true vale v in period 0.

In this setting, liquidity providers set $p_3 = v$. Assume that a pricing rule follows Eq.(1) with Assumption 1 in periods 1 and 2. Then, by Proposition 1, a pair of price coefficients should be in set C for preventing price manipulation. Otherwise the speculator undertakes price manipulation *even if he/she knows* $p_0 = v$.

Suppose that $v \neq p_0$ and $(\lambda, \mu) \in C$. Then the maximization problem for the speculator is

$$\max -p_1 x_1 - p_2 x_2 - v x_3$$

subject to $x_3 = -x_1 - x_2$. The first-order condition for this problem is

$$\begin{pmatrix} 2\mu & \beta\mu^2 - \mu + \lambda \\ \beta\mu^2 - \mu + \lambda & 2\mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v - p_0 \\ v - p_0 \end{pmatrix}.$$

Therefore

$$x_1 = x_2 = \frac{v - p_0}{\beta\mu^2 + \mu + \lambda},$$

which is the optimal solution of this problem because the objective function is concave. Then the equilibrium prices are

$$p_1 = p_0 + \frac{\mu(v - p_0)}{\beta\mu^2 + \mu + \lambda}, \qquad p_2 = v$$

and the payoff of the speculator is $(\beta\mu^2 + \lambda)(v - p_0)^2/(\beta\mu^2 + \mu + \lambda)^2$. Since $p_2 - p_1 = (\beta\mu^2 + \lambda)(v - p_0)/(\beta\mu^2 + \mu + \lambda)$, the price approaches v monotonically.

Thus a market price eventually coincides with the fundamental value if and only if $(\lambda, \mu) \in C$.

3.2 Round-trip strategies and time-independent price coefficients

This section exhibits a characterization of manipulation-proof price coefficients when we allow a more general class of round-trip strategies under the following assumption.

Assumption 2 *We assume that price functions satisfy* $U_n(q_n) = P_n(q_n) = \lambda q_n$ *with* $\lambda \in \mathbb{R}_+$.

Under Assumption 2, we seek the set of price coefficients that achieve $\pi(\boldsymbol{x}; N) < 0$ for *all* nonzero round-trip strategies in an N-period market. A round-trip strategy is said to be *nonzero* if $x_n \neq 0$ for some n.

Definition 2 *A pair of price coefficients* (λ, λ) *has the* β-manipulation-proof property *in an* N*-period market if it makes* $\pi(\boldsymbol{x}; N) < 0$ *for any nonzero round-trip strategy* $\boldsymbol{x} \in \mathbb{R}^N \backslash \{\boldsymbol{0}\}$.

Let $\boldsymbol{x}$ be a nonzero round-trip strategy in an N-period market. Then the market order in a period n is $q_n = x_n + \xi_n$. By Assumption 2, $\xi_n = \beta\lambda(x_{n-1} + \xi_{n-1})$. We can easily show that $\xi_n = (\beta\lambda)^{n-1}x_1 + \cdots + \beta\lambda x_{n-1}$. Hence the prices from a round-trip strategy $\boldsymbol{x}$ are denoted by

$$\boldsymbol{p} = \boldsymbol{p}_0 + \lambda \boldsymbol{D}\boldsymbol{x},$$

where $\boldsymbol{p} = (p_1, \cdots, p_N)^\top$, $\boldsymbol{p}_0 = (p_0, \cdots, p_0)^\top$, $\top$ is the transportation operator, and

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1+\beta\lambda & 1 & 0 & & 0 \\ 1+\beta\lambda+(\beta\lambda)^2 & 1+\beta\lambda & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \sum_{i=1}^{N}(\beta\lambda)^{i-1} & \sum_{i=1}^{N-1}(\beta\lambda)^{i-1} & \sum_{i=1}^{N-2}(\beta\lambda)^{i-1} & \cdots & 1 \end{pmatrix}.$$

Then the maximization problem for the speculator is written as follows:

$$\begin{aligned} &\max -\boldsymbol{x}\cdot\boldsymbol{p} \quad \text{s.t.} \quad \mathbf{1}\cdot\boldsymbol{x}=0 \\ \Leftrightarrow &\max -\lambda\boldsymbol{x}^\top\boldsymbol{D}\boldsymbol{x} \quad \text{s.t.} \quad \mathbf{1}\cdot\boldsymbol{x}=0, \end{aligned}$$

where $\mathbf{1}$ is the N-tuple of 1. Consider the following transformation of $\boldsymbol{D} = (d_{ij})$ for symmetrization: the matrix $\boldsymbol{A} = (a_{ij})$ is defined such that

$$a_{ij} = \begin{cases} (d_{ij} + d_{ji})/2 & (i \neq j) \\ d_{ij} & (i = j). \end{cases}$$

Then $\boldsymbol{x}^\top\boldsymbol{D}\boldsymbol{x} = \boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x}$.

I use the following proposition to obtain the result.

Proposition 6 (Debreu (1952), Theorem 4.) *Let $\boldsymbol{A}$ be a symmetric (N, N) matrix and $\boldsymbol{b} = (b_1, \cdots, b_N)^\top$ be an N-dimensional vector with b_1 being different from zero. Then $\boldsymbol{x}^\top\boldsymbol{A}\boldsymbol{x} > 0$ for every $\boldsymbol{x} \neq \mathbf{0}$ such that $\boldsymbol{b}\cdot\boldsymbol{x} = \mathbf{0}$ if and only if*

$$\begin{vmatrix} 0 & \boldsymbol{b}_r^\top \\ \boldsymbol{b}_r & \boldsymbol{A}_r \end{vmatrix} < 0$$

for all $r \in \{2, 3, \cdots, N\}$, where $\boldsymbol{A}_r$ is the (r, r) submatrix of $\boldsymbol{A}$ obtained by retaining only the first r rows and columns of $\boldsymbol{A}$ and $\boldsymbol{b}_r$ is the r-dimensional vector obtained by the first r elements of $\boldsymbol{b}$.

We obtain a characterization result.

Theorem 2 *Suppose that p_0 is sufficiently large. A pair of price coefficients $(\lambda, \lambda) \in \mathbb{R}^2_+$ has the β-manipulation-proof property in an N-period market if and only if λ satisfies $|\boldsymbol{C}_r| < 0$ for all $r = 2, 3, \cdots, N$, where*

$$\boldsymbol{C}_r = \begin{pmatrix} 0 & \mathbf{1}_r^\top \\ \mathbf{1}_r & \boldsymbol{A}_r \end{pmatrix}.$$

A pair of α-manipulation-proof price coefficients (λ, λ) may also have the β-manipulation-proof property. For example, consider the case $N = 3$. (See also Table 1.) $|\boldsymbol{C}_2| < 0$ is equivalent to $0 \leq \beta\lambda < 1$ and $|\boldsymbol{C}_3| < 0$ is equivalent to $0 \leq \beta\lambda < (-1+\sqrt{5})/2$. By Theorem 1, a pair of price coefficients (λ, λ) that satisfies $0 \leq \beta\lambda \leq 1/2$ has the α-manipulation-proof property. Since $1 < -1+\sqrt{5}$, there exists a pair of (nontrivial) α-manipulation-proof price coefficients that also satisfies the β-manipulation-proof property in the three-period market.

Table 1: In the three-period market, a price coefficient $\lambda \in [0, 1/(2\beta)]$ satisfies $0 \leq \beta\lambda < 1$ and $0 \leq \beta\lambda < -1+\sqrt{5}/2$. Hence a pair of α-manipulation-proof price coefficients (λ, λ) can have the β-manipulation-proof property.

N	Round-trip strategy	Manipulation-proof area
2	Any	$0 \leq \beta\lambda < 1$
3	Any	$0 \leq \beta\lambda < -1+\sqrt{5}/2$
Any	Simple	$0 \leq \beta\lambda \leq 1/2$

3.3 Time-dependent price coefficients

Assumption 3 *We assume that price functions satisfy*

$$U_n(q_n) = \lambda_n q_n, \qquad P_n(q_n) = \mu_n q_n$$

for all $n \in \{1, \cdots, N\}$.

3.3.1 Prevention of simple strategies

I consider the case $\lambda_n = \mu_n$. Theorem 1 leads to the following result.

Corollary 1 *Suppose that* $\lambda_n = \mu_n$ *for all* n. *Then* $0 \leq \lambda_n \leq 1/(2\beta)$ *for all* n *ensures that the speculator does not implement simple strategies.*

Proof. The market prices with the coefficients under a simple strategy $x > 0$ are written as

$$\begin{aligned} p_n &= p_{n-1} + \lambda_n q_n \\ &= p_0 + \sum_{k=1}^{n} \lambda_k \xi_k + \lambda_1 x \\ &\leq p_0 + \sum_{k=1}^{n} \overline{\lambda} |\xi_k| + \overline{\lambda} x, \end{aligned} \tag{10}$$

where $\overline{\lambda} = \sup\{\lambda_n\}$. Because $\xi_0 = \xi_1 = 0$, $\xi_2 = \beta\lambda_1 x$, and $\xi_n = \beta\lambda_{n-1}\xi_{n-1}$ for $n \geq 3$, we can easily check that $\xi_n > 0$ for all $n \geq 2$. Theorem 1 shows that, when $\lambda = \mu$,

$$0 \leq \lambda \leq \frac{1}{2\beta}$$

is a necessary and sufficient condition for the α-manipulation-proof property. Hence $\overline{\lambda} \leq 1/(2\beta)$ is sufficient for the speculator not to implement simple strategies. ■

3.3.2 Prevention of skimming strategies

This section considers the strategies in a given N-period market such that a strategy $\boldsymbol{x} = (x_1, \cdots, x_N)$ satisfies $x_k = x$ and $x_{k+1} = -x$ for some k, $1 \leq k < N$, where $x \in \mathbb{R}$. I do not impose any further condition for other x_k. We call such strategies *skimming strategies.*

Definition 3 *A $2N$-tuple of price coefficients $(\lambda_1, \cdots, \lambda_N, \mu_1, \cdots, \mu_N)$ has the* γ-manipulation-proof property *if, for all $x \in \mathbb{R}$ and $1 \leq k < N$, we obtain*

$$(p_{k+1} - p_k)x \leq 0,$$

where p_k and p_{k+1} are defined by Eq.(1) with Assumption 3.

The γ-manipulation-proof property is the property that prevents price manipulation with skimming strategies in an N-period market. If price coefficients in an N-period market do not have the γ-manipulation-proof property, then there exists $k \in \{1, \cdots, N-1\}$ such that the speculator wants to switch his/her strategy to a large size skimming strategy in periods k and $k+1$. Hence the δ-manipulation-proof property provides us with one criterion how much we have to restrict pricing rules in order to hold down the cancel effect.

Theorem 3 *A price coefficients vector $(\lambda_1, \cdots, \lambda_N, \mu_1, \cdots, \mu_N)$ has the γ-manipulation-proof property if and only if $X_k \equiv \mu_k + \mu_{k+1}$ and $Y_k \equiv \mu_k \mu_{k+1}$ satisfy*

$$0 \leq Y_k \leq \frac{1}{\beta}\Big(X_k - \lambda_k\Big) \quad and \quad 0 \leq Y_k \leq \frac{X_k^2}{4} \tag{11}$$

for all $1 \leq k < N$.

Proof. A market price takes the form $p_n = p_{n-1} + \mu_n q_n + (\lambda_{n-1} - \mu_{n-1}) q_{n-1}$. Suppose that $x_k = x$ and $x_{k+1} = -x$. Then

$$\begin{aligned} p_{k+1} - p_k &= \mu_{k+1}\Big(-x + \xi_{k+1}\Big) + (\lambda_k - \mu_k)\Big(x + \xi_k\Big) \\ &= x\Big(\lambda_k - \mu_k - \mu_{k+1} + \mu_k \mu_{k+1} \beta\Big) \\ &\quad + \underbrace{\mu_k \mu_{k+1} \xi_k \beta + (\lambda_{k-1} + \mu_{k-1})\Big(x_{k-1} + \xi_{k-1}\Big)}_{=Z_k}. \end{aligned}$$

The payoff is $(p_{k+1} - p_k)x = x^2(\lambda_k - \mu_k - \mu_{k+1} + \mu_k \mu_{k+1} \beta) + x Z_k$. If $\lambda_k - \mu_k - \mu_{k+1} + \mu_k \mu_{k+1} \beta > 0$, then the payoff can be positive by taking x to be sufficiently large. Hence $\lambda_k - \mu_k - \mu_{k+1} + \mu_k \mu_{k+1} \beta \leq 0$ is necessary, which can be written as

$$Y_k \leq \frac{1}{\beta}\Big(X_k - \lambda_k\Big), \tag{12}$$

where $X_k = \mu_k + \mu_{k+1}$ and $Y_k = \mu_k \mu_{k+1}$. Since $\mu_n \geq 0$ for all $1 \leq n \leq N$, it must be $X_k \geq 0$ and $Y_k \geq 0$. Furthermore, because $\{\mu_n\}$ are real numbers, the equation $t^2 - X_k t + Y_k = 0$ has real roots. Hence it must be that

$$X_k^2 - 4Y_k \geq 0. \tag{13}$$

Conversely, if Eq.(11) holds for all k, $1 \leq k < N$, then it ensures that $p_{k+1} - p_k \leq 0$ for all k. ■

Figure 8 represents a graphical image of this result. If β is large enough and $\lambda_k \approx \lambda_{k+1}$, the γ-manipulation-proof property requires either μ_k or μ_{k+1} to be small enough, especially when one of them is large. On the other hand, if β is small enough, the result implies that almost all coefficients have the γ-manipulation-proof property, which is consistent with our intuition.

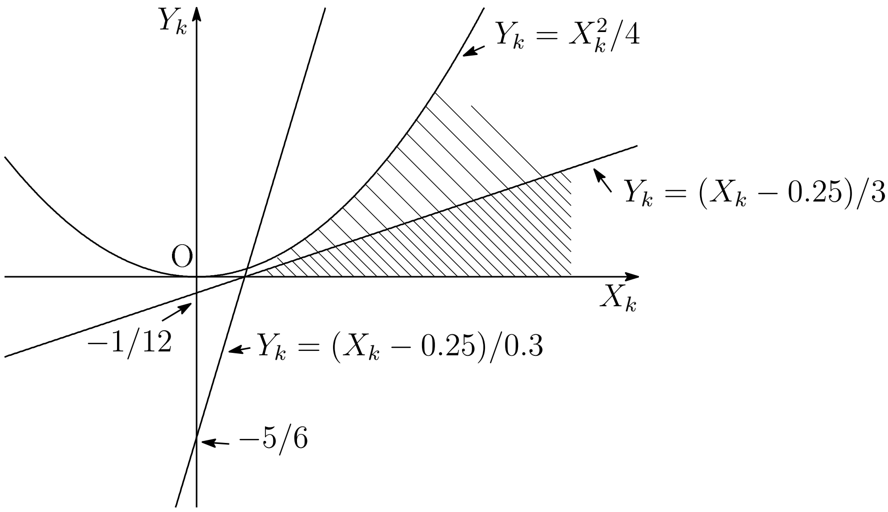

Figure 8: The figure sets $\lambda_k = 0.25$, $\beta = 3$ and 0.3. When β is large enough, which corresponds to the gentler line in the figure, Eq.(12) is effective. On the other hand, when β is small, which corresponds to the steeper one, Eq.(13) is effective.

When $\lambda_n = \lambda$ and $\mu_n = \mu$ for all n, Eq.(13) is trivially satisfied. In this case, Eq.(12) is equivalent to $\beta\mu^2 - 2\mu + \lambda \leq 0$, which is the same inequality in set C.

3.3.3 Application: the Kyle model

This section considers the Kyle (1985) model as an application of Corollary 1 and Theorem 3.

Now assume that

- The trading stock has a liquidation value v, which follows normal distribution $\mathcal{N}(v_0, \Sigma_0)$, and its true value is not public until the end of the market.

- The speculator knows the true value v privately in period 0.
- There is another trader, *trading crowds*, in the market, whose aggregate order placement in period n, say η_n, follows $\mathcal{N}(0, \sigma_c^2)$, which are mutually and serially independent of v.

The Kyle model assumes that liquidity providers know that the speculator knows the value v privately, whereas they only know its distribution. Competitive liquidity providers set market prices such that $\mathbf{E}[v|\{q_k\}_{k\leq n}] = p_n$, while the speculator trades with hiding his/her private information as much as possible.

Suppose that liquidity providers follow a pricing rule such that

$$p_n = p_{n-1} + \lambda_n q_n \quad (n = 1, \cdots, N) \tag{14}$$

with $p_0 = v_0$, where $\lambda_n \in \mathbb{R}_+$. This is the case of $\lambda_n = \mu_n$ for all n. Kyle (1985) shows that there exists a unieque sequence of equilibrium price coefficients $\{\lambda_k\}$ that constitutes Eq.(14). We consider the γ-manipulation-proof property for such an equilibrium sequence $\{\lambda_k\}$. In this case, Eq.(12) of Theorem 3 is equivalent to $-\lambda_{k+1}(1 - \lambda_k \beta) \leq 0$, which is equivalent to $\lambda_k \beta \leq 1$ or $\lambda_{k+1} = 0$. If $\{\lambda_k\}$ is an equilibrium price coefficients of the Kyle model, we know that $\lambda_k > 0$ and $\lambda_k < \lambda_{k-1}$. Hence we obtain the following statement.

Proposition 7 *The Kyle's equilibrium pricing rule has the γ-manipulation-proof property if and only if $\lambda_1 \beta \leq 1$.*

In fact, if $\lambda_n > 1/\beta$ for some n, then the speculator who follows an equilibrium strategy $\boldsymbol{x}^* = (x_1^*, \cdots, x_N^*)$ in the Kyle model is better to changes to strategy $\boldsymbol{x}'$ such that $x'_k = x^*_k$ for $k < n$, $x'_n = x$, $x'_{n+1} = -x$, and $x'_l = 0$ for all $l > n+1$ because

$$\mathbf{E}[(p_{n+1} - p_n)x_n] = \lambda_{n+1}\Big((\beta\lambda_n - 1)x^2 + \beta\xi_n x\Big).$$

Hence the speculator can gain a better profit by taking x to be large enough.

However, Corollary 1 implies that Proposition 7 is not sufficient to prevent other price manipulations.

Proposition 8 *The Kyle's equilibrium pricing rule has the α-manipulation-proof property only if* $0 \leq \lambda_N \leq 1/(2\beta)$.

Proof. Suppose that $\lambda_N > 1/(2\beta)$. The payoff of a simple strategy x is

$$\begin{aligned}(p_N - p_1)x &= x^2(\beta\lambda_2\lambda_1 + \cdots + \beta^{N-1}\lambda_N \cdots \lambda_1 - \lambda_N) \\ &> x^2\lambda_N\{\beta\lambda_N + \cdots + (\beta\lambda_N)^{N-1} - 1\}.\end{aligned}$$

Since $\lambda_N > 1/(2\beta)$, the speculator gets a positive payoff when N is sufficiently large. ■

These results imply that Kyle's equilibrium pricing strategy itself does not ensure the prevention of price manipulation. In other words, competitive pricing of liquidity providers does not ensure no mispricing.

However, if liquidity providers are aware of positive feedback traders and can estimate their trade size correctly, e.g., they know β, then they ignore $\{\xi_n\}$ in price formation because the order of positive feedback traders contains no information about v. In this case, the competitive pricing among liquidity providers can prevent price manipulation.

3.4 Price manipulation under risk

We have investigated price manipulation in the case where the speculator is risk-neutral. In this section, we consider a "risk-adjusted" α-manipulation-proof property: we evaluate the performance of price manipulation by the *Sharpe ratio*. In general, it is natural that a risk-adjusted property is more relaxed than a risk-neutral property, but this intuition is not right for the α-manipulation-proof property. Our risk-adjusted criterion still requires a pair of price coefficients to be α-manipulation-proof price coefficients.

As in Section 3.3.3, we introduce *trading crowds* as a noise trader. For notational convenience, we denote a random variable by "$\tilde{\ }$" above a letter and the expectation and the variance of a random variable by $\mathbf{E}$ and $\mathbf{V}$ respectively.

Assumption 4 *There are trading crowds in the market, whose aggregate* *order placement in period n, denoted by $\tilde{\eta}_n$, follows an identical and independent distribution with the expected value* 0 *and the variance σ_c^2.*

In this section, we evaluate the performance of price manipulation with round-trip strategies by the Sharpe ratio. Hubermann and Stanzl (2004) call it *quasi-arbitrage*. We here consider the quasi-arbitrage with *simple strategies*. Let $\tilde{\pi}(x_m; N) \equiv (\tilde{p}_N - \tilde{p}_1)x_m$ denote the payoff from simple strategy $x_m \in \mathbb{R}$ in an N-period market.

Definition 4 *A* quasi-arbitrage with simple strategies *is the price manipulation that satisfies*

$$\lim_{m\to\infty} \frac{\mathbf{E}[\tilde{\pi}(x_m; N)]}{\sqrt{\mathbf{V}[\tilde{\pi}(x_m; N)]}} = \infty.$$

Definition 5 *A pair of price coefficients (λ, μ) has the* δ-manipulation-proof property *if we obtain*

$$\sup_{N\in\mathbb{Z}_+} \left\{ \lim_{m\to\infty} \frac{\mathbf{E}[\tilde{\pi}(x_m; N)]}{\sqrt{\mathbf{V}[\tilde{\pi}(x_m; N)]}} \right\} < \infty, \tag{15}$$

where prices are defined by Eq.(1) with Assumptions 1.

Here is the main result in this section.

Theorem 4 *A pair of price coefficients (λ, μ) has the δ-manipulation-proof property if and only if it has the α-manipulation-proof property.*

I provide a sketch of proof for this proposition. By using the results in the proof of Theorem 1 (in particular, equations (A.6), (A.9), and (A.12) in the Appendix), we obtain a deterministic function $Z(N, \lambda, \mu, \beta)$ such that

$$\mathbf{E}[\tilde{\pi}(x_m; N)] = x_m^2 Z(N, \lambda, \mu, \beta).$$

(We have already seen a version of this Z as Eq.(7) in Theorem 1.) Proposition A.2 in the Appendix shows that, when $(\lambda, \mu) \neq (0, 0)$,

$$\mathbf{V}[\tilde{\pi}(x_m; N)] = x_m^2 \sigma_c^2 \sum_{k=1}^{N} \Big(F_k(\lambda, \mu, \beta) \Big)^2 > 0$$

with deterministic functions $\{F_k\}$, which are independently defined of x_m. Hence we obtain

$$\frac{\mathbf{E}[\tilde{\pi}(x_m; N)]}{\sqrt{\mathbf{V}[\tilde{\pi}(x_m; N)]}} = \frac{x_m Z(N, \lambda, \mu, \beta)}{\sigma_c \sqrt{\sum_{k=1}^{N} \Big(F_k(\lambda, \mu, \beta)\Big)^2}}.$$

Therefore $\sup Z(N, \lambda, \mu, \beta) \leq 0$ is a necessary and sufficient condition for the δ-manipulation-proof property. As we see in the proof of Theorem 1 in the Appendix, it is the equivalent condition for the α-manipulation-proof property.

This result is due the assumption that the noise is independently defined of x_m and β. When $Z(N, \lambda, \mu, \beta) > 0$ for some N, a quite large size of x_m brings the speculator a large payoff and makes the noise be relatively negligible.

4 Concluding remarks

This paper analyzed stock price manipulation exploiting positive feedback traders in a continuous auction market. The main results relate to the pricing rules of liquidity providers. To prevent price manipulation, we must restrict linear pricing rules to suppress the nonlinear cancel and rising effects. The obtained results are serious because it is true when we only considered simple classes of speculative strategies, i.e., simple strategies and skimming strategies. This is in sharp contrast to the result of Huberman and Stanzl (2004, Proposition 3), which says that any linear pricing rule satisfying $\lambda \leq 2\mu$ is sufficient to achieve no price manipulation when there is no noise trader even if we consider any round-trip strategy. Our results shed light on the relation between mispricing and the behavior of liquidity providers. We saw that liquidity providers can avoid mispricing if (i) their pricing rules are sufficiently small relative to a positive feedback magnitude, or (ii) they are aware of positive feedback traders. This result is contrast to the result of De Long et al (1990), which says that mispricing always occurs in equilibrium. I believe that the model is tractable and the results are easy to test empirically, which is left as future work.

A Appendix

A.1 Second-order linear difference equation

First, I introduce the theory of difference equations in order to prove Theorem 1.[8] A difference equation appearing in the model is defined by

$$q_{n+2} = Kq_{n+1} + Jq_n, \tag{A.1}$$

where $K, J \in \mathbb{R}$ and $n \in \mathbb{Z}_+ = \{1, 2, \cdots, \}$. Consider the equation

$$y^2 - Ky - J = 0. \tag{A.2}$$

Let φ_1 and φ_2 be the roots of Eq.(A.2). Suppose that $q_n = c_1\varphi_1^n + c_2\varphi_2^n$ for some c_1 and $c_2 \in \mathbb{R}$ and for all n, then

$$\begin{aligned}
&q_{n+2} - Kq_{n+1} - Jq_n \\
&= c_1\varphi_1^{n+2} + c_2\varphi_2^{n+2} - K(c_1\varphi_1^{n+1} + c_2\varphi_2^{n+1}) - J(c_1\varphi_1^n + c_2\varphi_2^n) \\
&= c_1\varphi_1^n(\varphi_1^2 - K\varphi_1 - J) + c_2\varphi_2^n(\varphi_2^2 - K\varphi_2 - J) \\
&= 0.
\end{aligned} \tag{A.3}$$

Therefore $q_n = c_1\varphi_1^n + c_2\varphi_2^n$ forms a solution of Eq.(A.1).

The solution of (A.1) is uniquely obtained by an *initial value* of Eq.(A.1). The model gives $(q_0, q_1) = (0, x)$, then c_1 and c_2 are determined uniquely by solving

$$\begin{pmatrix} 1 & 1 \\ \varphi_1 & \varphi_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

The roots of Eq.(A.2) is described as

$$\varphi_1 = \frac{K + \sqrt{D}}{2}, \quad \varphi_2 = \frac{K - \sqrt{D}}{2}, \tag{A.4}$$

[8] For more detail on difference equations see Elaydi (2005), for example.

where $D = K^2 + 4J$. Assume that $D \neq 0$. Then, $c_1 = x/\sqrt{D}$ and $c_2 = -x/\sqrt{D}$. Hence $q_n = x(\varphi_1^n - \varphi_2^n)/\sqrt{D}$. Assume conversely that $D = 0$. Then, $\varphi_1 = \varphi_2 = K/2$. In this case, the solution is $q_n = (K/2)^n(a_1 + na_2)$, where a_1 and a_2 are determined uniquely by an initial value. We can check it in the same way as in Eq.(A.3). When $q_0 = 0$ and $q_1 = x$, we obtain $q_n = xn(K/2)^{n-1}$.

A.2 Proof of Theorem 1

We provide a proof for the theorem in the case of simple strategy $x > 0$. (The argument is symmetric for the case $x < 0$ as long as p_0 is sufficiently large.) The simple strategy x generates the positive feedback traders' expected demands as follows:

$$\xi_n = \beta\mu q_{n-1} + \beta(\lambda - \mu)q_{n-2}, \tag{A.5}$$

which are equal to q_n when $n \in \{2, \cdots, N-1\}$. To shorten the notation, We denote $K = \beta\mu$, $J = \beta(\lambda - \mu)$, and $D = K^2 + 4J = (\beta\mu)^2 - 4\beta\mu + 4\beta\lambda$. Then, equation (A.5) represents a second-order linear homogeneous difference equation for $n \in \{2, \cdots, N-1\}$.

Case (i) $D > 0$

By using the technique in Section A.1, we obtain

$$\xi_n = \frac{x}{\sqrt{D}}\left(\varphi_1^n - \varphi_2^n\right)$$

for all $n \in \{1, \cdots, N-1\}$, where φ_1 and φ_2 are defined in Eq.(A.4). We first show that the expected payoff increases monotonically with N. Define

$$\begin{aligned} f_n &\equiv \varphi_1^n - \varphi_2^n \\ &= \left(\frac{K + \sqrt{D}}{2}\right)^n - \left(\frac{K - \sqrt{D}}{2}\right)^n. \end{aligned}$$

Lemma A.1 *When $D > 0$, we obtain $f_n > 0$ for all $n \in \mathbb{Z}_+$.*

Proof. Because $|\varphi_1|^2 - |\varphi_2|^2 = K\sqrt{D} \geq 0$, we have $|\varphi_1| \geq |\varphi_2|$. Hence $f_n \geq |\varphi_1|^n - |\varphi_2|^n \geq 0$ since $\varphi_1 > 0$. That $f_n = 0$ for some $n \in \mathbb{Z}_+$ is equivalent to $\varphi_1 = \varphi_2$, which implies $D = 0$. ■

The payoff function is

$$(p_N - p_1)x = x^2 \left\{ \frac{1}{\sqrt{D}} \left(\lambda \sum_{n=1}^{N-1} f_n + \mu f_N \right) - 2\mu \right\}. \tag{A.6}$$

By Lemma A.1, Eq.(A.6) increases monotonically with N.

Next we derive a necessary condition for the α-manipulation-proof price coefficients.

Lemma A.2 *Assume that $D > 0$. If (λ, μ) has the α-manipulation-proof property, then it must hold that $0 \leq \beta\lambda < 1$ and $0 \leq \beta\mu < 2$, both of which imply $\varphi_1 < 1$.*

Proof. Suppose that (λ, μ) has the α-manipulation-proof property. Then Eq.(A.6) and Lemma A.1 imply $\{\sum f_n\}$ must converge. Note that it implies $f_n \to 0$ as $n \to \infty$. Then,

$$\lambda \sum_{n=1}^{N-1} f_n + \mu f_N = \lambda \sum_{n=1}^{N-1} (\varphi_1^n - \varphi_2^n) + \mu(\varphi_1^N - \varphi_2^N).$$

By Lemma A.1, $f_1 = \varphi_1 - \varphi_2 > 0$. If $\varphi_1 = 1$, then $\{\sum f_n\}$ does not converge because $1 = \varphi_1 > |\varphi_2| \geq \varphi_2$. If $\varphi_1 > 1$, then $\varphi_2 > 1$ is required for convergence. However, $\{f_n\}$ does not converge in this case because

$$\begin{aligned} \varphi_1^n - \varphi_2^n &= (\varphi_1 - \varphi_2)(\varphi_1^{n-1} + \varphi_1^{n-2}\varphi_2 + \cdots + \varphi_2^{n-1}) \\ &> n\sqrt{D} \to \infty \quad (n \to \infty). \end{aligned}$$

Therefore the sequence $\{\sum f_n\}$ converges only if $\varphi_1 < 1$. When $D > 0$, it is equivalent to

$$\begin{aligned} &0 < \frac{K + \sqrt{D}}{2} < 1 \\ &\Leftrightarrow 0 < \sqrt{D} < 2 - K. \end{aligned}$$

Hence convergence of $\{\sum f_n\}$ requires $\beta\mu < 2$. Then

$$\begin{aligned} & D < 4 - 4K + K^2 \\ & \Leftrightarrow \beta\lambda < 1. \end{aligned}$$

Conversely, if $0 \leq \beta\lambda < 1$ and $0 \leq \beta\mu < 2$ hold, then we obtain $\varphi_1 < 1$. ■

The monotonicity of f_n leads to the following necessary and sufficient condition for the α-manipulation-proof coefficients.

Lemma A.3 *When $D > 0$. (λ, μ) has the manipulation-proof property if and only if*

$$0 \leq \beta\mu < 2, \quad \lambda - 2\mu + 2\beta\lambda\mu \leq 0. \tag{A.7}$$

Proof. Suppose that (λ, μ) has the α-manipulation-proof property. Then, by Lemma A.2, (λ, μ) satisfies $0 \leq \beta\lambda < 1$ and $0 \leq \beta\mu < 2$, and hence $\varphi_1 < 1$. Then we obtain $\sum \varphi_1^n < \infty$. Because $\varphi_1 = |\varphi_1| > |\varphi_2|$, $\varphi_1 < 1$ implies absolute convergence of $\sum_{n=1}^{\infty} f_n$. Thus

$$\begin{aligned} & \lim_{N\to\infty} \frac{1}{\sqrt{D}} \left(\lambda \sum_{n=1}^{N-1} (\varphi_1^n - \varphi_2^n) + \mu(\varphi_1^N - \varphi_2^N) \right) - 2\mu \\ & = \frac{\lambda}{\sqrt{D}} \left\{ \lim_{N\to\infty} \sum_{n=1}^{N-1} \varphi_1^n - \lim_{N\to\infty} \sum_{n=1}^{N-1} \varphi_2^n \right\} - 2\mu \\ & = \frac{\lambda}{\sqrt{D}} \left(\frac{\varphi_1}{1-\varphi_1} - \frac{\varphi_2}{1-\varphi_2} \right) - 2\mu \\ & = \frac{\lambda - 2\mu + 2\beta\lambda\mu}{1 - \beta\lambda}. \end{aligned} \tag{A.8}$$

Because Lemma A.1 leads to the payoff Eq.(A.6) increasing monotonically with N, the α-manipulation-proof property requires that $\lambda - 2\mu + 2\beta\lambda\mu \leq 0$. Next suppose that (λ, μ) satisfies Eq.(A.7). The inequality $\lambda - 2\mu + 2\beta\lambda\mu \leq 0$ is equivalent to $\lambda \leq (2\mu)/(1 + 2\beta\mu)$. Then $(2\mu)/(1 + 2\beta\mu) \uparrow 1/\beta$ as $\mu \uparrow \infty$. Hence $\lambda - 2\mu + 2\beta\lambda\mu \leq 0$ implies $\lambda < 1/\beta$ for all $\mu \geq 0$. Then we obtain $\varphi_1 < 1$, hence Eq.(A.8) is nonpositive. Because the payoff Eq.(A.6) increases monotonically with N, it implies that (λ, μ) has the α-manipulation-proof property. ■

Case (ii) $D = 0$

The previous section showed that

$$\xi_n = nx\left(\frac{K}{2}\right)^{n-1}$$

for all $n \geq 2$. Thus

$$(p_N - p_1)x = x^2\left\{\lambda\left(\sum_{n=1}^{N-1} n\left(\frac{K}{2}\right)^{n-1}\right) + \mu\left(N\left(\frac{K}{2}\right)^{N-1}\right) - 2\mu\right\}. \quad \text{(A.9)}$$

Lemma A.4 *Assume that $D = 0$. (λ, μ) has the α-manipulation-proof property if and only if Eq.(A.7) holds, which implies*

$$0 \leq \mu \leq \frac{7-\sqrt{17}}{4\beta}, \qquad 0 \leq \lambda \leq \frac{7-\sqrt{17}}{(9-\sqrt{17})\beta}.$$

Proof. We can see that $nx(K/2)^{n-1} \geq 0$ for all $n \geq 2$ and $nx(K/2)^{n-1} = 0$ if and only if $K = 0$ and $J = 0$, which imply $\mu = \lambda = 0$. In that case, (λ, μ) has the α-manipulation-proof property and satisfies $0 \leq \beta\mu < 2$ and Eq.(A.7). Hereafter we assume $nx(K/2)^{n-1} > 0$ for all $n \geq 2$. Because Eq.(A.9) increases monotonically with N in that case, the α-manipulation-proof property requires μ to ensure $0 \leq \beta\mu < 2$. In that case, by using the theorem on integration term by term, we obtain

$$\lambda\sum_{n=1}^{\infty} n\left(\frac{K}{2}\right)^{n-1} = \frac{\lambda}{(1-\beta\mu/2)^2}.$$

Hence the condition for (λ, μ) to have the α-manipulation-proof property is $0 \leq \beta\mu < 2$ and

$$\begin{aligned}
&\lim_{N\to\infty} \lambda \left(\sum_{n=1}^{N-1} n \left(\frac{K}{2}\right)^{n-1} \right) + \mu \left(N \left(\frac{K}{2}\right)^{N-1} \right) - 2\mu \\
&= \frac{\lambda}{(1-\beta\mu/2)^2} - 2\mu \\
&\leq 0.
\end{aligned}$$

The last inequality is equivalent to $\lambda - 2\mu + 2\beta\lambda\mu \leq 0$. This inequality and the assumption $\beta\mu^2 - 4\mu + 4\lambda = 0$ imply $\mu \leq (7-\sqrt{17})/(4\beta)$ or $(7+\sqrt{17})/(4\beta) \leq \mu$. The inequalities $0 \leq \mu < 2/\beta$ and $\lambda \leq 2\mu/(2\beta\mu+1)$ lead to $0 \leq \mu \leq (7-\sqrt{17})/(4\beta)$ and $0 \leq \lambda \leq (7-\sqrt{17})/\{(9-\sqrt{17})\beta\}$. ■

Case (iii) $D < 0$

In that case, we obtain

$$\begin{aligned}
\xi_n &= \frac{x}{i\sqrt{D'}} \{\varphi_1^n - \varphi_2^n\} \\
&= \frac{x}{i\sqrt{D'}} \left\{ \left(\frac{K+i\sqrt{D'}}{2}\right)^n - \left(\frac{K-i\sqrt{D'}}{2}\right)^n \right\}
\end{aligned} \tag{A.10}$$

for all $n \geq 2$, where $D' = -D > 0$ and $i = \sqrt{-1}$. Note that $D < 0$ implies $J < 0$. Let $J' = -J > 0$. By using Euler's formula, we denote φ_1 and φ_2 in polar form.

$$\varphi_1 = \sqrt{J'}(\cos\theta + i\sin\theta) = \sqrt{J'}e^{i\theta}, \qquad \varphi_2 = \overline{\varphi_1} = \sqrt{J'}e^{-i\theta}.$$

By using De Moivre's theorem, Eq.(A.10) is equivalent to

$$\xi_n = \frac{2x}{\sqrt{D'}} \left(\sqrt{J'}\right)^n \sin(n\theta) \tag{A.11}$$

for $n \geq 2$ with θ satisfying $\xi_1 = x$. Then, $\cos\theta = K/(2\sqrt{J'})$ and $\sin\theta = \sqrt{D'}/(2\sqrt{J'})$. Because $K/(2\sqrt{J'}) \geq 0$ and $\sqrt{D'}/(2\sqrt{J'}) > 0$, it must be $\theta \in (0, \pi/2]$. The payoff of the speculator from a simple strategy x is

$$x^2 \left(\frac{2\lambda}{\sqrt{D'}} \sum_{n=1}^{N-1} (\sqrt{J'})^n \sin(n\theta) + \frac{2\mu}{\sqrt{D'}} (\sqrt{J'})^N \sin(N\theta) - 2\mu \right). \tag{A.12}$$

Lemma A.5 *Assume that $D < 0$. If (λ, μ) has the α-manipulation-proof property, then it must hold that*

$$\beta\mu - 1 < \beta\lambda, \quad 0 \leq \beta\lambda < 1, \quad \lambda - 2\mu + 2\beta\lambda\mu \leq 0. \tag{A.13}$$

Proof. The α-manipulation-proof property requires the payoff Eq.(A.12) to be nonpositive for all N. It requires Eq.(A.12) to be bounded hence it must be that $\sqrt{J'} < 1$, which is equivalent to $\beta\mu - 1 < \beta\lambda$. Because $D = (\beta\mu)^2 - 4\beta\mu + 4\beta\lambda$ and $\beta > 0$, $D < 0$ implies $\beta\lambda < 1$. Because $J' < 1$ implies absolute convergence of $\sum_{n=1}^{\infty} \xi_n$, we obtain, from Eq.(A.10), that

$$\begin{aligned} \sum_{n=1}^{\infty} \xi_n &= \frac{x}{i\sqrt{D'}} \left\{ \frac{\varphi_1}{1 - \varphi_1} - \frac{\varphi_2}{1 - \varphi_2} \right\} \\ &= \frac{x}{1 - \beta\lambda}. \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{N \to \infty} x^2 \left(\frac{2\lambda}{\sqrt{D'}} \sum_{n=1}^{N-1} (\sqrt{J'})^n \sin(n\theta) + \frac{2\mu}{\sqrt{D'}} (\sqrt{J'})^N \sin(N\theta) - 2\mu \right) \\ &= x^2 \left(\frac{\lambda}{1 - \beta\lambda} - 2\mu \right) \\ &= x^2 \left(\frac{\lambda - 2\mu + 2\beta\lambda\mu}{1 - \beta\lambda} \right). \end{aligned}$$

Because we have obtained $\beta\lambda < 1$, the α-manipulation-proof property requires $\lambda - 2\mu + 2\beta\lambda\mu \leq 0$. ■

Note that the necessary condition proposed in Lemma A.2 to Lemma A.4 can be replaced by Eq.(A.13). To see this, we consider the functions $\lambda = F(\mu) = -\beta\mu^2/4 + \mu$ and $\lambda = G(\mu) = \mu - 1/\beta$. We can easily check that $F(\mu) > G(\mu)$ when $0 \leq \beta\mu < 2$. Moreover, we know from Lemma A.4 that (λ, μ) satisfies $\beta\mu^2 - 4\mu + 4\lambda \geq 0$ and $\lambda \leq 2\mu/(2\beta\mu + 1)$ when $0 \leq \mu \leq (7 - \sqrt{17})/(4\beta)$. Because $(7 - \sqrt{17})/(4\beta) < 1/\beta = G^{-1}(0)$, we can replace the condition $0 \leq \beta\mu < 2$ with $\beta\mu - 1 < \beta\lambda$ when $D \geq 0$. I summarize the argument as the following proposition.

Proposition A.1 *If (λ, μ) has the manipulation-proof property, then (λ, μ) satisfies the following inequalities.*

$$\beta\mu - 1 < \beta\lambda, \ 0 \leq \beta\lambda < 1, \ \lambda - 2\mu + 2\beta\lambda\mu \leq 0. \tag{A.14}$$

In particular, when $D \geq 0$, (λ, μ) has the manipulation-proof property if and only if (λ, μ) satisfies

$$\lambda - 2\mu + 2\beta\lambda\mu \leq 0, \ 0 \leq \mu < \frac{7 - \sqrt{17}}{4\beta}, \ 0 \leq \lambda < \frac{7 - \sqrt{17}}{(9 - \sqrt{17})\beta}.$$

Theorem 1 follows from Proposition A.1.

A.3 Failure of payoff monotonicity

Unfortunately, Eq.(A.14) is not a sufficient condition for the α-manipulation-proof property because payoff monotonicity fails when $D < 0$.

Lemma A.6 *Suppose that (λ, μ) satisfies $D < 0$ and*

$$\beta\mu - 1 < \beta\lambda \quad \text{and} \quad 0 \leq \beta\lambda < 1.$$

Then, there are infinitely many N such that $p_N > p^* := \lim_{n\to\infty} p_n$.

Proof. $p_N > p^*$ is equivalent to

$$\frac{\mu}{\lambda}\xi_N > \sum_{n=N}^{\infty} \xi_n. \tag{A.15}$$

We show that there are infinitely many N that satisfy Eq.(A.15). The assumption $\beta\mu - 1 < \beta\lambda$ implies $|\varphi_1| = \sqrt{J'} < 1$. Then $\sum_{n=N}^{\infty} \xi_n$ absolutely converges. Hence

$$\begin{aligned}
\sum_{n=N}^{\infty} \xi_n &= \frac{x}{i\sqrt{D'}} \left\{ \sum_{n=N}^{\infty} (\varphi_1^n - \varphi_2^n) \right\} \\
&= \frac{x}{i\sqrt{D'}} \left\{ \sum_{n=N}^{\infty} \varphi_1^n - \sum_{n=N}^{\infty} \varphi_2^n \right\} \\
&= \frac{x}{i\sqrt{D'}} \left\{ \frac{\varphi_1^N - \varphi_2^N - \varphi_1^N\varphi_2 + \varphi_1\varphi_2^N}{(1-\varphi_1)(1-\varphi_2)} \right\}.
\end{aligned}$$

Note that $(1-\varphi_1)(1-\varphi_2) = 1 - \beta\lambda$. By using Eq.(A.11) and the fact $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, we obtain

$$\varphi_1^N - \varphi_2^N - \varphi_1^N\varphi_2 + \varphi_1\varphi_2^N = 2i(\sqrt{J'})^N \Big(\sin(N\theta) - \sqrt{J'} \sin\big((N-1)\theta\big) \Big).$$

Again by using Eq.(A.11),

$$\begin{aligned}
&\frac{\mu}{\lambda}\xi_N > \sum_{n=N}^{\infty} \xi_n \\
\Leftrightarrow \quad &(\mu(1-\beta\lambda) - \lambda)\sin(N\theta) > -\lambda\sqrt{J'}\sin\big((N-1)\theta\big).
\end{aligned} \tag{A.16}$$

Because $-\lambda\sqrt{J'} \leq 0$ and $D = K^2 + 4J < 0$ imply $J' > 0$, it must be that $-\lambda\sqrt{J'} = 0$ if and only if $\lambda = 0$ and $\mu \neq 0$ when $\lambda = 0$. Therefore there are two possibilities: $\mu(1-\beta\lambda) - \lambda > 0$ or $\mu(1-\beta\lambda) - \lambda < 0$. If the former holds, then we select N such that $\sin(N\theta) > 0$ and $\sin((N-1)\theta) > 0$; if the latter holds, we select N such that $\sin(N\theta) < 0$ and $\sin((N-1)\theta) > 0$. There are infinitely many such N because $\theta \neq 0$. ■

Lemma A.6, together with Figure 6, suggests that there exists N' and $2 \leq k^* \leq N'$ such that $(p_{k^*} - p_1)x \geq (p_n - p_1)x$ for all $2 \leq n \leq N'$ in the N'-period market. If this case happens, the problem is that k^* is determined by (λ, μ); there may be many (λ, μ) satisfying Eq.(A.13) and $D < 0$ that define different k^*s in a common N'-period market.

A.4 Numerical properties on the areas S_1 and S_2

Let S_1 denote the area of (λ, μ) defined by Eq.(A.7) and $D > 0$; S_2 denote the area of (λ, μ) defined by Eq.(A.13).

Lemma A.7 *S_1 has a positive measure in $\mathbb{R}_+^2$ for all $\beta > 0$.*

Proof. The proof of Lemma A.4 tells us, when $\lambda = 2\mu/(2\beta\mu + 1)$, that the equation $\beta\mu^2 - 4\mu + 4\lambda = 0$ has three roots, $\mu = 0, (7 - \sqrt{17})/(4\beta)$, and $(7 + \sqrt{17})/(4\beta)$. Eq.(A.7) means $\lambda \leq 2\mu/(2\beta\mu + 1)$. $D > 0$ requires $(4 - \beta\mu^2)/4 < \lambda$. Then

$$\begin{aligned}\int_0^{\frac{7-\sqrt{17}}{4\beta}} \frac{2\mu}{2\beta\mu + 1} - \frac{4\mu - \beta\mu^2}{4} d\mu &= \int_0^{\frac{7-\sqrt{17}}{4\beta}} \frac{1}{4}\beta\mu^2 - \mu + \frac{1}{\beta} - \frac{1}{\beta}\left(\frac{1}{2\beta\mu + 1}\right) d\mu \\ &= \frac{1}{192\beta^2}\left(115 - 5\sqrt{17} - 96\log\left(\frac{9 - \sqrt{17}}{2}\right)\right) \\ &> \frac{1}{192\beta^2} \cdot \frac{88}{10} \left(= \frac{11}{240\beta^2}\right).\end{aligned} \tag{A.17}$$

■

Lemma A.8 *If (λ, μ) has the α-manipulation-proof property, then (λ, μ) must be in B, where*

$$B = \{(\lambda, \mu) \mid 0 \leq \beta\lambda < 1, \quad 0 \leq \beta\mu < 2, \quad \lambda \leq 2\mu\}. \tag{A.18}$$

Proof. Lemmas A.2, A.4, A.5 require (λ, μ) to satisfy $0 \leq \beta\lambda < 1$ and $0 \leq \beta\mu < 2$. Consider $N = 2$ and the payoff $(p_2 - p_1)x = (\beta\mu^2 - 2\mu + \lambda)x^2$. If $\lambda > 2\mu$, the payoff is positive even if $\beta = 0$. ■

Lemma A.9 *$S_1/B \approx 0.026$ and $S_2/B \approx 0.41$.*

Proof. It is easy to show that the area B equals $7/(4\beta^2)$. From Eq.(A.17), we obtain

$$\frac{1}{192\beta^2}\cdot\frac{88}{10} < \frac{1}{192\beta^2}\left(115-5\sqrt{17}-96\log\left(\frac{9-\sqrt{17}}{2}\right)\right) < \frac{1}{192\beta^2}\cdot\frac{89}{10}.$$

Dividing each side by $7/(4\beta^2)$ gives

$$0.0261 < S_1/B < 0.0264.$$

The solution of $\mu - 1/\beta = 2\mu/(2\beta\mu+1)$ is $\mu = (3+\sqrt{17})/(4\beta)$. Because $1 < (3+\sqrt{17})/4 < 2$, the area S_2 equals

$$\begin{aligned}\int_0^{\frac{3+\sqrt{17}}{4\beta}} \frac{2\mu}{2\beta\mu+1}d\mu - \frac{(-1+\sqrt{17})^2}{32\beta^2} &= -\frac{1}{\beta}\left[\frac{1}{2\beta}\log(2\beta\mu+1)-\mu\right]_0^{\frac{3+\sqrt{17}}{4\beta}} - \frac{9-\sqrt{17}}{16\beta^2} \\ &= \frac{1}{16\beta^2}\left\{3+5\sqrt{17}-8\log\left(\frac{5+\sqrt{17}}{2}\right)\right\}.\end{aligned}$$

Then we obtain

$$\frac{1}{16\beta^2}\cdot\frac{1147}{100} < \frac{1}{16\beta^2}\left\{3+5\sqrt{17}-8\log\left(\frac{5+\sqrt{17}}{2}\right)\right\} < \frac{1}{16\beta^2}\cdot\frac{1148}{100}.$$

Dividing each side by $7/(4\beta^2)$ gives

$$0.4096 < S_2/B < 0.4100.$$

■

A.5 Proof of Theorem 4

Lemma A.10 *Suppose that the speculator implements a simple strategy x. Then, for all $2 \leq n \leq N-1$, there exists a random valuable $\tilde{y}_n$ such that $\tilde{\xi}_n = \xi_n + \tilde{y}_n$ and is independent of x.*

Proof. We prove the statement by mathematical induction. When $n = 2$, $\tilde{\xi}_2 = \beta(\tilde{p}_1 - \tilde{p}_0) = \beta\mu x + \beta\mu\tilde{\eta}_1$. Hence $\tilde{y}_2 = \beta\mu\tilde{\eta}_1$. Suppose that $\tilde{\xi}_k = \xi_k + \tilde{y}_k$ for all $2 \leq k \leq n$. Then $\tilde{\xi}_{n+1} = \beta(\mu\tilde{q}_n + (\lambda - \mu)\tilde{q}_{n-1})$. Since $\tilde{q}_k = \tilde{\xi}_k + \tilde{\eta}_k$ and $\xi_k = \beta(\mu\xi_{k-1} + (\lambda - \mu)\xi_{k-2})$, it is true that

$$\begin{aligned}\tilde{\xi}_{n+1} &= \beta\Big(\mu(\xi_n + \tilde{y}_n + \tilde{\eta}_n) + (\lambda - \mu)(\xi_{n-1} + \tilde{y}_{n-1} + \tilde{\eta}_{n-1})\Big) \\ &= \xi_{n+1} + \beta\Big(\mu(\tilde{y}_n + \tilde{\eta}_n) + (\lambda - \mu)(\tilde{y}_{n-1} + \tilde{\eta}_{n-1})\Big).\end{aligned}$$

Hence we put $\tilde{y}_{n+1} = \beta(\mu(\tilde{y}_n + \tilde{\eta}_n) + (\lambda - \mu)(\tilde{y}_{n-1} + \tilde{\eta}_{n-1}))$. Since

$$\begin{aligned}\tilde{y}_{n+1} &= \beta(\mu(\tilde{y}_n + \tilde{\eta}_n) + (\lambda - \mu)(\tilde{y}_{n-1} + \tilde{\eta}_{n-1})) \\ &= \beta(\mu\tilde{y}_n + (\lambda - \mu)\tilde{y}_{n-1}) + \beta(\mu\tilde{\eta}_n + (\lambda - \mu)\tilde{\eta}_{n-1}),\end{aligned}$$

we see that $\tilde{y}_n$ is independent of x when we define $\tilde{y}_0 = \tilde{y}_1 = 0$. ■

By the same reasoning, we obtain the following lemma.

Lemma A.11 *Suppose that the speculator implements a simple strategy x. Then, for all $n \in \{2, \cdots, N-1\}$, there exists a deterministic sequence $\{E_1(n,\beta), \cdots, E_{n-1}(n,\beta)\}$ such that*

$$\tilde{y}_n = \sum_{j=1}^{n-1} E_j(n,\beta)\tilde{\eta}_j.$$

By using the lemmas, we obtain the following result.

Proposition A.2 *There exists a deterministic sequence $\{F_1(\lambda,\mu,\beta), \cdots, F_N(\lambda,\mu,\beta)\}$ such that*

$$\mathbf{V}[\tilde{\pi}(x;N)] = x^2\sigma_c^2 \sum_{k=1}^{N} \Big(F_k(\lambda,\mu,\beta)\Big)^2 \geq 0,$$

where equality holds if and only if $(\lambda,\mu) = (0,0)$.

Proof. By using Lemma A.10, we obtain

$$\begin{aligned}\tilde{p}_N - \tilde{p}_1 =& (\lambda - 2\mu)x + \lambda \sum_{k=2}^{N-1} \xi_k + \mu \xi_N \\ &+ (\lambda - \mu)\tilde{\eta}_1 + \lambda \sum_{k=2}^{N-1} \tilde{\eta}_k + \mu \tilde{\eta}_N + \lambda \sum_{k=2}^{N-1} \tilde{y}_k + \mu \tilde{y}_N.\end{aligned}$$

By using Lemma A.11, we obtain

$$\lambda \sum_{k=2}^{N-1} \tilde{y}_k + \mu \tilde{y}_N = \lambda \sum_{k=2}^{N-1} \sum_{j=1}^{k-1} E_j(k,\beta)\tilde{\eta}_j + \mu \sum_{j=1}^{N-1} E_j(N,\beta)\tilde{\eta}_j.$$

Hence the result follows from

$$\begin{aligned}&(\lambda - \mu)\tilde{\eta}_1 + \lambda \sum_{k=2}^{N-1} \tilde{\eta}_k + \mu \tilde{\eta}_N + \lambda \sum_{k=2}^{N-1} \tilde{y}_k + \mu \tilde{y}_N \\ =& \underbrace{\Big(\lambda - \mu + \lambda \sum_{k=2}^{N-1} E_1(k,\beta) + \mu E_1(N,\beta)\Big)}_{=F_1(\lambda,\mu,\beta)} \tilde{\eta}_1 + \sum_{j=2}^{N-1} \underbrace{\Big(\lambda + \lambda \sum_{k=j+1}^{N-1} E_j(k) + \mu E_j(N)\Big)}_{=F_j(\lambda,\mu,\beta)} \tilde{\eta}_j + \mu \tilde{\eta}_N\end{aligned}$$

with putting $\mu = F_N(\lambda,\mu,\beta)$. ■

We see $Z(N,\lambda,\mu,\beta)$ in Eq.(A.6), Eq.(A.9), and Eq.(A.12) when $D > 0$, $D = 0$, and $D < 0$ respectively.

References

1. Barberis, N., and A. Shleifer (2003), "Style investing", *Journal of Financial Economics* 68: 161–199.

2. Chakraborty, A., and B. Yilmaz (2004a), "Informed manipulation", *Journal of Economic Theory* 114: 132–152.

3. ——— (2004b), "Manipulation in market order models", *Journal of Financial Markets* 7: 187–206.

4. Debreu, G. (1952), “Definite and semidefinite quadratic forms”, *Econometrica* 20: 295–300.

5. De Long, J., A. Shleifer, L. Summers and R. Waldmann (1990), “Positive feedback investment strategies and destabilizing rational speculation”, *Journal of Finance* 45: 379–395.

6. Elaydi, S. (2005), *An Introduction to Difference Equations.* Springer.

7. Glosten, L., and P. Milgrom (1985), “Bid, ask and transaction prices in a specialist market with heterogeneously informed traders”, *Journal of Financial Economics* 13: 71–100.

8. Hong, H., and J. Stein (1999), “A unified theory of underreaction, momentum trading, and overreaction in asset markets”, *Journal of Finance* 54: 2143–2184.

9. Huberman, G., and W. Stanzl (2004), “Price manipulation and quasi-arbitrage”, *Econometrica* 72: 1247–1275.

10. Jarrow, R. (1992), “Market manipulation, bubbles, corners, and short squeezes”, *Journal of Financial and Quantitative Analysis* 27: 311–336.

11. Kyle, A. (1985), “Continuous auctions and insider trading”, *Econometrica* 53: 1315–1335.

12. Shiller, R. (2008), *The Subprime Solution.* Princeton University Press.

13. Soros, G. (2003), *The Alchemy of Finance.* Wiley.