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Measure

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# Estimation of Default Probability under Real Probability Measure

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## Abstract

We propose a model of deriving the default probability of a company from the market data. We use the risk neutral pricing method directly and obtain the approximation formulae of real world default probability. The data analysis shows that the default probabilities under real measure are estimated smaller than those under risk neutral measure as expected.

## 1 Introduction

We shall consider a way to obtain the default probability of some specific company or the average default probability of given rating classes. If we rely only on the time of default for estimating the probabilities, usual statistical method won't work for each individual company's default probability as the targeting company has not defaulted yet. These probabilities may be obtained from the finance data via sophisticated methods. We may, however, derive these impliedly from the interest spread observed at the market, which are obtained under the equivalent martingale measure though. The methods for deriving implied probability have several advantages over the regular statistical methods. Here, we are to consider two probability spaces, one with "real" probability and the other with equivalent martingale measure. We shall consider the probability of default of certain events in both space and their estimators. We probably must be careful distinguishing the probability of the events and their estimators. We shall, however, be sloppy about these matters, hoping that the difference will be clear from the context.

Since, other estimators estimated statistically are considered in the "real" probability space, we need to transform estimated implied default probabilities into the ones in real probability. The real probability here may be subjective or objective. We note that under certain conditions, implied default probability under the martingale measure and real measure is equal (cf. Takahashi[2011]).

Moreover, we don't know what mean by the probability of default under the equivalent martingale measure or default probability at the risk neutral world. And the resulting default probability is given under the martingale measure. Therefore, it is not appropriate to compare these probabilities directly with the ones obtained under the real probability (statistically obtained probability).

The average default probabilities for each rating classes may be derived impliedly as for the individual cases. Here, we may raise the question. If we can compare these probabilities, as one is under the real, the other is under the martingale measure. To make the comparison possible, we need to convert the probability under the martingale measure to the real probability or vice versa. The resulting default probability, however, is given under the equivalent martingale measure and it cannot be compared to the statistically obtained probability.

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Therefore, it would be better to have a method converting implied probabilities (under the risk neutral world) to the ones under the "real" probability measure. The recent paper by Ross [2013], discusses the way to construct the real measure from the martingale measure. His method may be applicable to our problem, we shall develop different approach which is more or less confined to the estimation of default probabilities.

## 2 Main Results

Let  $(\Omega, \mathcal{F}, \Pr)$  be a probability space. And we shall consider discrete time normal model. We let  $T = [0, 1, 2, \dots, T]$  be a set of time points at which trading is taken place. Let,  $\{x_1, x_2, \dots, x_t\}$ ,  $0 \leq t \leq T$  are independent and identically distributed random variables with  $N(\mu, \sigma^2)$  under the "real" probability measure  $\Pr$  and this will be a source of randomness in our economic model. Let  $\{\mathcal{F}_t, t = 0, 1, \dots, T\}$  be an increasing sequence of sigma algebra for which  $\mathcal{F}_t \in \sigma\{x_1, x_2, \dots, x_t\}$ ,  $t = 1, 2, \dots, T$ ,  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ , and  $\mathcal{F}_T = \mathcal{F}$ . We also denote  $\Pr^{(t)}$ ,  $t = 1, \dots, T$  the restriction of  $\Pr$  to  $\mathcal{F}_t$ .

We set  $r(t)$  be a risk free interest rate for the period  $[t, t+1)$ ,  $t = 0, 1, \dots, T-1$ . We assume that the random evolution of the  $r(t)$  is given by the following stochastic difference equation,

$$r(t+1) - r(t) = \alpha(t, \omega) + \beta(t, \omega)x_t \quad t = 0, 1, \dots, T-1, \quad (1)$$

where  $r(0) = r_0$  is a given constant.

In the economy we consider, there are default free government bond and risky corporate bond. We suppose, except their credibility, they are equivalent. Let,  $P^*(t, s)$  and  $P(t, s)$  be the time  $t$  price of zero coupon riskless (government) bond maturing at  $s$  ( $t < s < T$ ) and risky (corporate) bond with the same maturity respectively. First, we consider the riskless bond and we assume that,

$$P^*(t, t) = 1 \quad \text{for all } t = 0, 1, \dots, T, \quad (2)$$

$$P^*(t, s) \geq 0 \quad \text{for all } t, \quad (3)$$

$$P^*(t+1, s) - P^*(t, s) = \mu^*(t, s)P^*(t, s) + \sigma^*(t, s)P^*(t, s)x_t. \quad (4)$$

We shall also suppose that the market is frictionless. Then, it is straight forward to show by the method of Vasicek[1977], the necessary and sufficient condition for the no arbitrage opportunity in the market is the existence of the function  $\lambda(t)$  depends only on  $t$  for which

$$\frac{\mu^*(t, s) - (1 + r(t))}{\sigma^*(t, s)} = \lambda(t) \quad \text{for all } 0 \leq t \leq s \quad (5)$$

The equivalent martingale measure  $Q$  is now defined by

$$dQ^{(t)} = \exp \left\{ \sum_{i=1}^t \lambda_i x_i - \frac{1}{2} \sum_{i=1}^t \lambda_i^2 \right\} d\Pr^{(t)}, \quad t = 1, \dots, T \quad (6)$$

(cf. Heath, Jarrow and Morton[1992]), where we have set  $\lambda_i = \lambda(i)$ ,  $x_i = x(i)$ ,  $i = 1, \dots, t$ . We note that  $\sum_{i=1}^t x_i$  is distributed with  $N(\sum_{i=1}^t \lambda_i, t)$  under  $Q$ . Needless to say that  $\left\{ \frac{P^*(t, T)}{B(t)}, \mathcal{F}_t, t = 0, 1, \dots, T \right\}$  is a martingale under  $Q$ , where  $B(t) = \prod_{s=0}^{t-1} (1 + r_s)$ .

It follows that

$$P^*(t, T) = B(t)E^Q \left\{ \frac{1}{B(T)} \middle| \mathcal{F}_t \right\} \quad (7)$$

Now, to consider the pricing of risky bond issued by company  $k$ , we let  $\tau = \tau_k$  be the time of the default of that company ( $k = 1, \dots, K$ ).

$$\begin{aligned}
\Pr\{\tau > t\} &= E^{\Pr}\{I_{[\tau > t]}\} = \int_{[\tau > t]} d\Pr \\
&= \int_{[\tau > t]} \frac{d\Pr^{(t)}}{dQ^{(t)}} dQ \\
&= E^Q \left\{ I_{[\tau > t]} \exp \left\{ - \sum_{i=1}^t \lambda_i x_i + \frac{1}{2} \sum_{i=1}^t \lambda_i^2 \right\} \right\} \\
&= E^Q \left\{ I_{[\tau > t]} \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) - \frac{1}{2} \sum_{i=1}^t \lambda_i^2 \right\} \right\} \\
&= e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} E^Q \left\{ I_{[\tau > t]} + I_{[\tau > t]} \left[ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right] \right\} \\
&= e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} Q\{\tau > t\} + e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} E^Q \left\{ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right\} - e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} Rem
\end{aligned} \tag{8}$$

where,  $E^{\Pr}$  and  $E^Q$  are expectations under  $\Pr$  and  $Q$  respectively, and

$$Rem = E^Q \left\{ I_{[\tau \leq t]} \left[ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right] \right\} \tag{9}$$

We estimate  $\Pr\{\tau > t\}$  by

$$\begin{aligned}
\tilde{\Pr}\{\tau > t\} &= e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} Q\{\tau > t\} \\
&\quad + e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} E^Q \left\{ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right\}
\end{aligned} \tag{10}$$

Then,

$$|\tilde{\Pr}\{\tau > t\} - \Pr\{\tau > t\}| \leq e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} Rem, \tag{11}$$

where

$$Rem = E^Q \left\{ I_{[\tau \leq t]} \left[ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right] \right\} \tag{12}$$

In order to control the remainder term, the trick here is to replace the event  $\{\tau > t\}$  by  $\{\tau \leq t\}$  the second term inside the expectation sign in the equation (9). This comes from the observation that for the most of the interesting application, the value of  $t$  is less than 10(years) and the probability of the event  $\{\tau \leq t\}$  under  $Q$  is substantially small. It follows that we may consider the third term on the right most side of (9) as a remainder. We next use Schwartz inequality,

$$\begin{aligned}
|Rem| &\leq E^Q \left\{ I_{[\tau \leq t]} \left[ \left| \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right| \right] \right\} \\
&\leq \sqrt{Q\{\tau \leq t\}} \sqrt{E^Q \left\{ \exp \left\{ -2 \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 2 \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} + 1 \right\}} \\
&= \sqrt{Q\{\tau \leq t\}} \sqrt{e^{2 \sum_{i=1}^t \lambda_i^2} - 2e^{\frac{1}{2} \sum_{i=1}^t \lambda_i^2} + 1}
\end{aligned} \tag{13}$$

Since

$$E^Q \left\{ \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right\} = \exp \left\{ \frac{1}{2} \sum_{i=1}^t \lambda_i^2 \right\} - 1 \quad (14)$$

It follows that,

$$\Pr\{\tau > t\} = e^{-\frac{1}{2} \sum_{i=1}^t \lambda_i^2} \left[ Q\{\tau > t\} + \exp \left\{ \frac{1}{2} \sum_{i=1}^t \lambda_i^2 \right\} - 1 + Rem \right] \quad (15)$$

where,

$$|Rem| \leq \sqrt{Q\{\tau \leq t\}} \sqrt{e^{2 \sum_{i=1}^t \lambda_i^2} - 2e^{\frac{1}{2} \sum_{i=1}^t \lambda_i^2} + 1}. \quad (16)$$

For the further simplification, we suppose  $\lambda_i = \lambda$  for all  $i \geq 1$ . It follows

$$\Pr\{\tau > t\} = e^{-\frac{t}{2} \lambda^2} \left[ Q\{\tau > t\} + \exp \left\{ \frac{t \lambda^2}{2} \right\} - 1 - Rem \right] \quad (17)$$

and

$$|Rem| \leq \sqrt{Q\{\tau \leq t\}} \sqrt{e^{2t\lambda^2} - 2e^{\frac{t}{2}\lambda^2} + 1}. \quad (18)$$

### 3 Numerical Analysis

In this section, we provide numerical analysis of our model. We use market data for calculating the default probability  $\Pr\{\tau > t\}$ . Using corporate bond price data of individual companies, we calculate representative values of  $\Pr\{\tau > t\}$  for each rating classes.

#### 3.1 The Data

Data for this study consist of overnight call rates, government bonds and corporate bonds traded in the markets between January 2003 and December 2012. We use the uncollateralized overnight call rate published by Bank of Japan. Figure 1 shows the historical overnight call rate and the yields of government bond between January 2003 and December 2012.

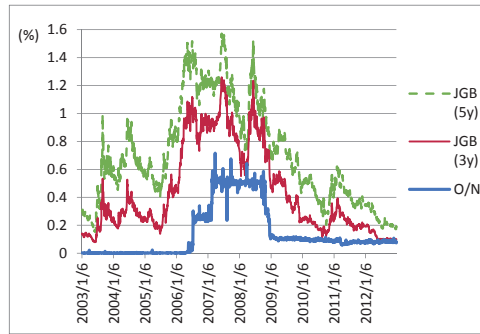


Figure 1: Historical JGB and O/N interest rate

*Note:* Historical overnight call rate and the yields of government bond between January 2003 and December 2012. The blue solid line shows the historical overnight call rate. Both red solid line and green dotted line show the JGB yields, the red solid line is for JGBs whose remaining periods are 3 years and the dotted line is for JGBs whose remaining periods are 5 years.

| Rating Class | Total Number | Number Per Day | Average Yield | Average Term |
|--------------|--------------|----------------|---------------|--------------|
| AA           | 2475         | 891            | 0.0102        | 6.29         |
| A            | 3362         | 795            | 0.00992       | 3.74         |
| BBB          | 1908         | 334            | 0.0147        | 2.93         |

Table 1: Summary measures for corporate bonds

*Note:* The column labeled "Total Number" displays the total number of corporate bonds traded in the market between January 2003 and December 2012. The column labeled "Number Per Day" displays the average number of corporate bonds per day traded in the market. The column labeled "Average Yield" displays the average corporate bond yield. The column labeled "Average Term" displays the average term to maturity of corporate bonds.

We also use straight bonds issued by AA, A, BBB credit rating companies. (We excluded convertible bonds and subordinated bonds.) We reference ratings information provided by Rating and Investment Information, Inc.(R&I). In Table 1, we summarize the properties of corporate bonds.

### 3.2 Implementing the historical volatility

The bond price data  $\{ p(t, s), t = 0, 1, \dots, s \}$  are observed in the market. Using  $N + 1$  bond price data  $p(t - N, s), p(t - (N - 1), s), \dots, p(t, s)$ , we calculate the  $N$  days historical volatility  $\sigma(t, s)$  by the following equation

$$\sigma(t, s) = \sqrt{\frac{250}{N-1} \sum_{i=0}^{N-1} (R(t-i, s) - \bar{R}_t)^2} \quad (19)$$

$$R(t, s) = \ln P(t, s) - \ln P(t-1, s), \quad (20)$$

where

$$\bar{R}_t = \frac{1}{N} \sum_{i=0}^{N-1} R(t-i, s).$$

We set  $N = 20$  in the analysis below.

Furthermore,  $\mu^*(t, s)$ ,  $1 + r(t, s)$  are obtained from the government bond price data  $\{ p^*(t, s), t = 0, 1, \dots, s \}$  and the overnight call rate, respectively. The market price of risk  $\lambda(t)$  is then calculated as follows:

$$\lambda(t) = \frac{\mu^*(t, s) - (1 + r(t, s))}{\sigma(t, s)} \quad (21)$$

for each  $t$ .

### 3.3 Strategy

We extract data in two different appointed periods from the historical price data of individual bonds. The term to maturity  $u = s - t$  is the time remaining on a bond  $P(t, s)$ 's life. We set the following two periods;

Period1 : 2years <  $u$  < 4years  
Period2 : 4years <  $u$  < 6years

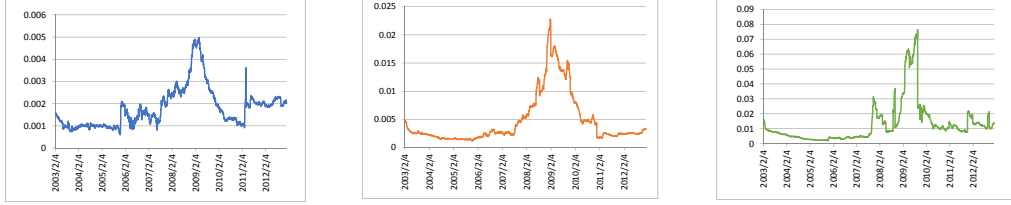


Figure 2: Historical Average Spreads for 3 Rating Classes

Note: The average spreads of the corporate bonds and government bonds whose terms to the expiration are 0.5 ~ 1.5 years. The left graph is for AA rating class, the center is for A, the right is for BBB.

In this paper, we define "yield spread"  $\Upsilon$  as yield spread of corporate bond over government bond.

$$\Upsilon = -\frac{1}{s-t} \log \left[ \frac{P(t, s)}{P^*(t, s)} \right]$$

We show the arithmetic means of yield spreads for three rating classes in Figure 2

We find from Figure 2 that there are several big fluctuations on spreads. This may show the possibility of existence of outlier data.

Now, We shall compute implied survival probability  $Q\{\tau > t\}$ . We use Duffie & Singleton[1999] framework to derive  $Q\{\tau > t\}$ . Under their framework, we assume  $\delta_t \times P(\tau-, t)$  is recovered at the time of default, where  $\delta_t$  is the recovery rate and  $P(\tau-, t)$  is the price of corporate bond just before default. Assuming the hazard function to be nonrandom and the recovery rate  $\delta_t$  at default to be constant<sup>1</sup>, implied survival function is

$$Q\{\tau > t\} = \left( e^{-t\Upsilon} \right)^{\frac{1}{1-\delta}} = \left( \frac{P(0, t)}{P^*(0, t)} \right)^{\frac{1}{1-\delta}}$$

### 3.4 Results of data analysis

Now, we use the model developed in Section 2 to estimate the default probabilities under real measure. The results are shown for 2 period groups.

First, We compute the market price of risk  $\lambda_t$  for each period groups. Table 2 is the statistics of estimated market price of risk  $\lambda_t$ . Figure 3 shows the historical  $\lambda_t$  estimate for each period group.

We observe from Table 2 and Figure 3 that the estimates of market price of risk  $\lambda_t$  vary widely. The mean and the variance of the estimates for Period 1 are larger than those for Period 2. And the distribution is right-skewed for Period 1, but left-skewed for Period 2.

Next, we compute  $\tilde{Pr}\{\tau > t\}$  and  $Q\{\tau > t\}$  for each period groups. Figure 4 displays calculating results of  $\tilde{Pr}\{\tau > t\}$  and  $Q\{\tau > t\}$  for 2 period groups.

As expected, the default probabilities under real measure are smaller than those under risk neutral measure. The market price of risk  $\lambda_t$  has a large effect on the calculation of  $Pr\{\tau > t\}$

<sup>1</sup>As mentioned in Chapter1, under these conditions, implied default probability under the martingale measure and real measure is equal. Although these assumptions seem contradictory to our model, we shall adopt the framework to compute rough estimate.

|          | Mean  | SD    | Kurtness | Skewness |
|----------|-------|-------|----------|----------|
| Period 1 | 0.430 | 0.192 | 0.687    | 0.488    |
| Period 2 | 0.328 | 0.125 | -0.381   | 0.287    |

Table 2: Statistics for the market price of risk  $\lambda_t$

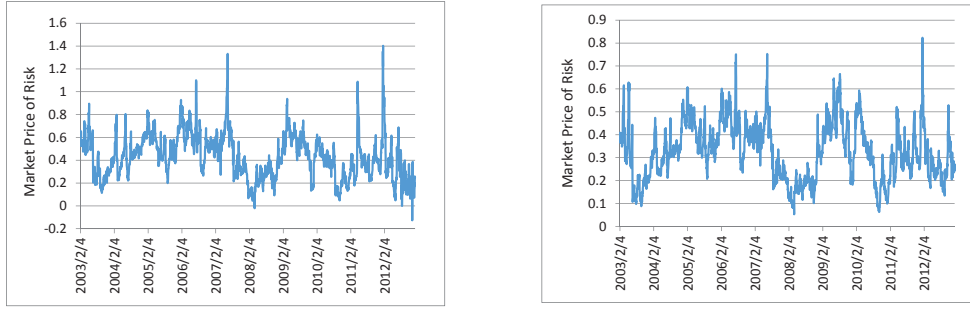


Figure 3: The Market Price of Risk  $\lambda_t$  for Period1 and Period2.

Note: The graphs of the historical market price of risk  $\lambda_t$  in Eq.(21). The left graph is for period1 and the right is for period2.

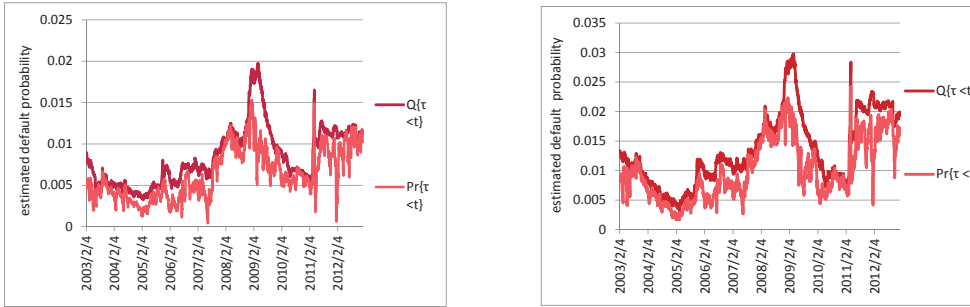


Figure 4: Estimates of  $\hat{Pr}\{\tau \leq t\}$  and  $Q\{\tau \leq t\}$  for the Period 1 and 2.

Note: The graphs of the historical  $\hat{Pr}\{\tau > t\}$ (the orange line) and  $Q\{\tau > t\}$ (the deep red line) in Eq.(17). The left graph is for period1 and the right is for period2.

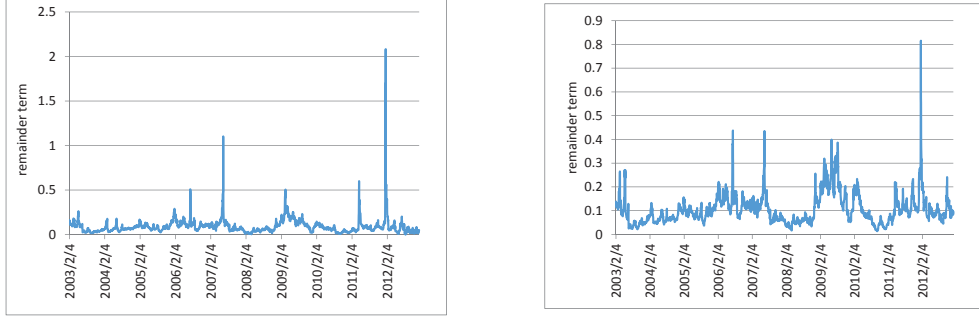


Figure 5: The upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  using Schwartz inequality.

Note: The graphs of the historical upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$ . The left graph is for period1 and the right is for period2.

We next evaluate the remainder term  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  by Schwartz inequality. Using Eq. (13), we calculate the upper bounds of  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$ . Figure 5 shows the historical upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  for each period group.

### 3.5 Further Model Modification

To evaluate remainder terms more precisely, we apply Hölder's inequality to (9), instead of Schwartz inequality.

For some  $p, q \in [1, \infty)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we rewrite (16) as follows.

$$\begin{aligned} |Rem| &\leq E^Q \left\{ I_{[\tau \leq t]} \left| \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right| \right\} \\ &\leq (Q(\tau \leq t))^{\frac{1}{p}} \left( E^Q \left\{ \left| \exp \left\{ - \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} - 1 \right|^q \right\} \right)^{\frac{1}{q}} \end{aligned} \quad (22)$$

Since

$$E^Q \left\{ \exp \left\{ -a \sum_{i=1}^t \lambda_i (x_i - \lambda_i) \right\} \right\} = \exp \left\{ \frac{a^2}{2} \sum_{i=1}^t \lambda_i^2 \right\},$$

for some nonnegative integer  $a$ , we can compute the remainder term.

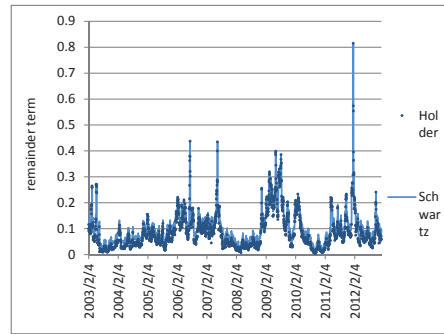
We analyze for  $q = 2, 3, \dots, 10$ . We select a set of  $p$  and  $q$ , which minimizes  $|Rem|$  in (22).

Figure 6 shows the remainder term changes when  $q$  changes from 2 to 7. The minimum  $|Rem|$  values are attained when  $q = 2 \sim 6$ . Figure 7 shows the upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  using Hölder's inequality. The remainder terms using Hölder's inequality become smaller than those using Schwartz inequality, but there is little difference between them.



**Figure 6:** The remainder term changes when  $q$  changes from 2 to 7.

*Note:* The graphs of the remainder term  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  using Hölder's inequality when  $q$  changes from 2 to 7. The left graph is for 2004/2/4 - 2004/2/26 and the right is for 2007/9/5-2007/9/28.



**Figure 7:** The upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  using Hölder's inequality.

*Note:* The graphs of the historical upper bound calculations of the remainder terms  $\exp\{-\frac{t}{2}\lambda^2\}|Rem|$  for period2.

## 4 Concluding Remarks

We present a model of deriving the default probability of a company from the market data. We use the risk neutral pricing method directly and obtain the approximation formulae of real world default probability. By using our model, we may compare the statistical default probability and the default probability calculated from the market data. From the data analysis, the default probabilities under real measure are estimated smaller than those under risk neutral measure as expected. Since the market price of risk  $\lambda(t)$  plays a significant role in our model,  $\lambda(t)$  estimates have great influence on the estimating results.

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