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A modified multilevel Monte Carlo method for European style options valuation

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Abstract

This article proposes a modified multilevel Monte Carlo (MMLMC) method and tests the method in terms of variance estimation. Many MMLMC estimators can be constructed under the MMLMC framework and therefore we can find the finest estimator, which has the lowest variance, in numerical experiments. The numerical results show that many of the MMLMC estimators perform better than both the original multilevel Monte Carlo (MLMC) estimator and the standard Monte Carlo estimator and that thinning some of the levels in the MLMC estimator improves the estimate of variance for European vanilla and Lookback options pricing.

Keywords: Monte Carlo; multilevel Monte Carlo; variance reduction; option pricing

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1. Introduction

The Monte Carlo method is already a useful computational tool in finance; however, its computational complexity becomes too large for achieving the required accuracy. [3] proposed a multilevel Monte Carlo (MLMC) method to reduce computational complexity and this study tests the MLMC method for pricing European style options. The MLMC method has been studied in recent years (e.g., variance reduction: see [5] and [8], Quasi Monte Carlo: [1],

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discretization scheme: [2], survey: [4]). We consider a stochastic differential equation (SDE) of the form

$$dS_t = \mu\left(S_t, t\right) dt + \sigma\left(S_t, t\right) dW_t, \qquad 0 \leq t < T, \tag{1}$$

where $S_t \in \mathbb{R}^m$, $S_0 = s$ is given, $T < \infty$, $W_t \in \mathbb{R}^d$ is a standard Brownian motion, and $\mu : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are drift and volatility coefficients, respectively. Let P be a discounted payoff function. In options pricing, $E[P(S_T)]$means the option price. We denote $t_0 = 0$ and $t_D = T$ and divide the interval $[0, T]$ in D subintervals of equal lengths, that is, $[t_0, t_1], [t_1, t_2], \ldots, [t_D - 1, t_D]$; $\Delta t \equiv t_n - t_{n-1} = T/D$ for any $1 \leq n \leq D$. Then the discretization of $\{S_t\}_t$ using an Euler scheme with Δt is given by

$$\widehat{S}_{t_{n+1}} - \widehat{S}_{t_n} = \mu\left(\widehat{S}_{t_n}, t_n\right) \Delta t + \sigma\left(\widehat{S}_{t_n}, t_n\right) \Delta W_{t_n}, \quad n = 0, 1, \ldots, D-1.$$

Where $\Delta W_{t_n} = W_{t_n} - W_{t_{n-1}}$. Under a standard Monte Carlo (SMC) method, we can compute Y, that is, the approximation of $E[P(S_T)]$

$$Y = N^{-1} \sum_{i=1}^{N} P\left(\widehat{S}_T^{(i)}\right),$$

where N is the number of simulation paths. Set $D = M^L$. Under the MLMC method, the option price is constructed by

$$\mathrm{E}[\widehat{P}_L] = \mathrm{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathrm{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], \tag{2}$$

where each P_ℓ is the approximation of $P(S_T)$ on level ℓ; $\widehat{P}_\ell$ means the discretization of $P(S_T)$ with a time step $h_\ell = T/M^\ell$. The coarsest level and the finest level are zero and L, respectively. The MLMC method uses all levels from zero to L. The MLMC estimator is uniquely constructed by

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

where

$$\widehat{Y}_\ell = \begin{cases} N_0^{-1} \sum_{i=1}^{N_0} \widehat{P}_0^i, & (\ell = 0), \\ N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^i - \widehat{P}_{\ell-1}^i\right), & (0 < \ell \leq L). \end{cases}$$

Note that $\widehat{Y}_0$ is the estimator of $E[P_0]$ using N_0 simulation paths and that each $\widehat{Y}_\ell$ is the estimator of $E[P_\ell - P_{\ell-1}]$ using N_ℓ paths for $\ell = 1, 2, \ldots, L$. We organize this paper as follows. In Section 2, we propose a modified multilevel Monte Carlo (MMLMC) method that allows the construction of multiple estimators. In Section 3, we illustrate the numerical performance of the MMLMC method for two European style options valuation. The final section discusses a future direction for research in this area.

2. Modified multilevel Monte Carlo method

2.1. Modified multilevel Monte Carlo estimators

We can rewrite the equation (2) as follows.

$$\mathrm{E}[\widehat{P}_L] = \mathrm{E}[\widehat{P}_{\ell_0}] + \sum_{j=1}^{F} \mathrm{E}[\widehat{P}_{\ell_j} - \widehat{P}_{\ell_{j-1}}], \tag{3}$$

where $\ell_F = L$ and $\ell_0 \geq 0$. Therefore, we can construct $2^{L-\ell_0-1}$ MMLMC estimators.

$$\ddot{Y} = \tilde{Y}_{\ell_0} + \sum_{j=1}^{F} \tilde{Y}_{\ell_j,\ell_{j-1}},$$

where

$$\tilde{Y}_{\ell_0} = N_{\ell_0}^{-1} \sum_{i=1}^{N_{\ell_0}} \widehat{P}_{\ell_0}^i,$$

and

$$\tilde{Y}_{\ell_j,\ell_{j-1}} = N_{\ell_j}^{-1} \sum_{i=1}^{N_{\ell_j}} \big(\widehat{P}_{\ell_j}^i - \widehat{P}_{\ell_{j-1}}^i\big), \quad 0 < j \leq F.$$

The variance of MMLMC estimators is

$$V[\ddot{Y}] = N_{\ell_0}^{-1} V_{\ell_0} + \sum_{j=1}^{F} N_{\ell_j}^{-1} V_{\ell_j,\ell_{j-1}},$$

where V_{ℓ_0} or $V_{\ell_j,\ell_{j-1}}$ is the variance of a single sample at each level.

2.2. Discussion

The mean squared error (MSE) of the MLMC or MMLMC estimators is given by

$$\mathrm{MSE}_{MLMC} = \mathrm{E}[(\widehat{Y} - \mathrm{E}[P])^2] = \mathrm{V}[\widehat{Y}] + (\mathrm{E}[\widehat{Y}] - \mathrm{E}[P])^2, \quad (4)$$

$$\mathrm{MSE}_{MMLMC} = \mathrm{E}[(\ddot{Y} - \mathrm{E}[P])^2] = \mathrm{V}[\ddot{Y}] + (\mathrm{E}[\ddot{Y}] - \mathrm{E}[P])^2, \quad (5)$$

where the first term on the right-hand side in (4), (5) is the variance of the estimator and the second term is the square of its bias due to discretization. [3] proves the MLMC complexity theorem. It claims that the computational complexity to attain $MSE < \epsilon^2$ is reduced from $O(\epsilon^{-3})$ to $O(\epsilon^{-2}(\ln \epsilon)^2)$ for a simple case.

Theorem 2.1 (Giles[3]). *Let P denote a function of the solution of SDE (1) for a given Brownian path $W(t)$, and, let $\widehat{P}_\ell$ denote the corresponding level ℓ of numerical approximation using a numerical discretization with time step $h_\ell = M^{-\ell}T$.*
If there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, \gamma, c_1, c_2, c_3$ such that

i) $|E[\widehat{P}_\ell - P]| \leq c_1 h_\ell^\alpha$,

ii) $E[\widehat{Y}_\ell] = \begin{cases} E[\widehat{P}_0], & \ell = 0, \\ E[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0, \end{cases}$

iii) $V[\widehat{Y}_\ell] \leq c_2 {N_\ell}^{-1} h_\ell^\beta$,

iv) C_ℓ, the computational complexity of $\widehat{Y}_\ell$, is bounded by

$$C_\ell \leq c_3 N_\ell h_\ell^{-1},$$

then, there exists a positive constant c_4 such that for any $\epsilon < e^{-1}$ there are values L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has a mean-square-error with bound

$$MSE \equiv E\left[\left(\widehat{Y} - E[P]\right)^2\right] < \epsilon^2$$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4\epsilon^{-2}, & \beta > 1, \\ c_4\epsilon^{-2}(\ln\epsilon)^2, & \beta = 1, \\ c_4\epsilon^{-2-(\gamma-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Proof. See Giles [3]. □

If the theorem holds, both the variance and the square of bias error in (4) have the same upper bound, $\epsilon^2/2$ (See the proof in Giles [3]). In comparison to the SMC method, the MLMC method reduces the variance, while leaving the bias unchanged. Variance reduction leads to reduction in computational complexity.

Similar to the MLMC complexity theorem in [3], if we use levels $\ell_0, \ell_1, \ldots, \ell_F$ and not levels $0, 1, \ldots, L$, we understand that the MMLMC version of the complexity theorem holds. We can prove the MMLMC complexity theorem on the basis of the proof procedure of the MLMC complexity theorem in [3] (See [6]). This means that both the variance and the square bias error of each MMLMC estimator have the same upper bound, $\epsilon^2/2$. If we assume (ii) in the complexity theorem in [3], the MLMC estimator is an unbiased estimator. Then, due to (2) and (3), each MMLMC estimator is an unbiased and $\mathrm{E}[\widehat{P}_L] = E[\widehat{Y}] = E[\ddot{Y}]$. Therefore, the bias errors of the MMLMC and MLMC estimators are the same. However, $V[\ddot{Y}]$ and $V[\widehat{Y}]$ are not always the same except in a special case; for example, if $\ell_0 = 0$ and $\ell_j - \ell_{j-1} = 1, j = 1, 2, \ldots, F$, the MMLMC estimator coincides with the MLMC estimator. Here, our focus is the performance comparison of the three methods (MMLMC, MLMC, and SMC) in terms of variance estimation.

3. Numerical experiments

In Section 3.1, we discuss the optimal number of simulation paths for the MMLMC method. In Section 3.2, we test the MMLMC estimators for European vanilla and Lookback options pricing. We compare the MMLMC method with the MLMC method and compare the MMLMC method with the SMC method in terms of variance reduction.

3.1. Optimal number of simulation paths

Let a computational complexity

$$C = \sum_{j=0}^{F} N_{\ell_j} M^{\ell_j} = \sum_{j=0}^{F} N_{\ell_j} T/h_{\ell_j}. \tag{6}$$

We regard $V[\ddot{Y}] = N_{\ell_0}^{-1} V_{\ell_0} + \sum_{j=1}^{F} N_{\ell_j}^{-1} V_{\ell_j,\ell_{j-1}}$ as a partially differentiable function of $N_{\ell_j}, j = 1, 2, \ldots, F$. We denote $V_{\ell_j,\ell_{j-1}} = V_{\ell_j}$, since there is no possibility of misunderstanding. Then the optimal number of paths for each level, minimizing the variance of each MMLMC estimator, are given by $\ddot{N}^*_{\ell_j}$ as follows.

Theorem 3.1. *For $\epsilon > 0$, the optimal number of simulation paths that achieve $V[\ddot{Y}] < \epsilon^2/2$, are given by*

$$\ddot{N}_{\ell_j} = \left\lceil 2\epsilon^{-2} \sqrt{V_{\ell_j} h_{\ell_j}} \left(\sum_{i=0}^{F} \sqrt{V_{\ell_i}/h_{\ell_i}} \right) \right\rceil, \quad 0 \le j \le F, \tag{7}$$

where $\lceil n \rceil$ is the least integer greater than or equal to n. Furthermore, for fixed computational complexity $C^* > 0$, the optimal number of simulation paths that minimize the variance of the corresponding MMLMC estimator are given by

$$\ddot{N}^*_{\ell_j} = \left\lceil \frac{C^* \sqrt{V_{\ell_j} h_{\ell_j}}}{\sum_{i=0}^{F} T \sqrt{V_{\ell_i}/h_{\ell_i}}} \right\rceil, \quad 0 \le j \le F. \tag{8}$$

Proof. To minimize $V[\ddot{Y}] = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j}$, we apply the Lagrange Multipliers method. We create the Lagrange equation as follows:

$$L := \mathcal{L}(N_{\ell_0}, N_{\ell_1}, \ldots, N_{\ell_F}) = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j} - \lambda \left(C - \sum_{j=0}^{F} N_{\ell_j} T/h_{\ell_j} \right).$$

Set the partial derivative $L_{N_{\ell_0}}, L_{N_{\ell_1}}, \dots, L_F$ equal to zero,

$$L_{N_{\ell_j}} = -N_{\ell_j}^{-2} V_{\ell_j} + \lambda T / h_{\ell_j} = 0, \quad j = 0, 1, \dots, F.$$

Thereby,

$$N_{\ell_j} = \sqrt{\frac{V_{\ell_j} h_{\ell_j}}{\lambda T}}, \quad j = 0, 1, \dots, F. \tag{9}$$

If $V[\ddot{Y}] < \epsilon^2/2$, it holds that

$$V[\ddot{Y}] = \sum_{j=0}^{F} N_{\ell_j}^{-1} V_{\ell_j} = \sum_{j=0}^{F} \sqrt{\frac{\lambda T}{V_{\ell_j} h_{\ell_j}}} V_{\ell_j} < \epsilon^2/2. \tag{10}$$

Due to (9) and (10), we set

$$N_{\ell_j} = 2\epsilon^{-2} \sqrt{V_{\ell_j} h_j} \left(\sum_{i=0}^{F} \sqrt{V_{\ell_i} / h_{\ell_i}} \right), \quad 0 \leq j \leq F. \tag{11}$$

Therefore, we get $\ddot{N}_j = [N_{\ell_j}]$. Due to (6) and (11)

$$2\epsilon^{-2} \sum_{i=0}^{F} \sqrt{V_{\ell_i} / h_{\ell_i}} = \frac{C}{\sum_{j=0}^{F} T \sqrt{V_{\ell_j} / h_{\ell_j}}}.$$

If we fix $C = C^*$, we get

$$\ddot{N}_{\ell_j}^* = \left[\frac{C^* \sqrt{V_{\ell_j} h_{\ell_j}}}{\sum_{i=0}^{F} T \sqrt{V_{\ell_i} / h_{\ell_i}}} \right], \quad 0 \leq j \leq F.$$

□

Note that if the MMLMC estimator coincides with the MLMC estimator, (7) is equal to the optimal simulation times used in Section 5 of [3].

3.2. Numerical results

We set $M = 2$, $\ell_0 = 0$, $\ell_F = 8$, and $C^* = 1,000,000 \times 2^8$. Therefore, we can test 128 types of MMLMC estimators. We present each result for

$$dS_t = 0.03 S_t dt + 0.3 S_t dW_t, \quad 0 \leq t < T,$$

where $S_0 = 1$ and $T = 1$. We note that the results of the SMC estimator are based on $1,000,000$ simulation paths and 2^8 time-steps. A European vanilla put and a Lookback put options payoffs at time T are $\max(K - S_T, 0)$ and $\max_{0\leq t\leq T} S_t - S_T$, respectively. We set the strike price $K = 1$. The numerical procedure is as follows. First, we set the initial $\ddot{N}_{\ell_j} = 10^5$. To calculate the optimal $\ddot{N}_{\ell_j}$, we run the MMLMC method using the initial paths. Second, we run the MMLMC method using the optimal paths and estimate the variance of the estimator. We repeat this procedure for all the MMLMC estimators.

Figure 1-2 show that more than 110 types of MMLMC estimators attain lower variance than that of the MLMC estimator.

- European vanilla option

In comparison to the MLMC estimator, the finest MMLMC estimator, which has the lowest variance and is $\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$, achieves variance reduction of about 20 %. The MMLMC estimators reduce variance by 88-96 % of the SMC estimator.

estimator	estimator construction	levels	variance
MMLMC (finest)	$\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$	0, 3, 5, 8	6.83E-10
MMLMC (coarsest)	$\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{8,0}$	0, 8	2.10E-9
MLMC	$\widehat{Y} = \sum_{\ell=0}^{8} \widehat{Y}_\ell$	0,1, ... , 8	8.53E-10
SMC	Y	-	1.88E-8

Table 1: comparison of variances: European vanilla option

estimator	$\ddot{N}^*_{\ell_0}$	$\ddot{N}^*_{\ell_1}$	$\ddot{N}^*_{\ell_2}$	$\ddot{N}^*_{\ell_3}$	$\ddot{N}^*_{\ell_4}$
MMLMC (finest)	9.8E7	-	-	7.4E7	-
MMLMC (coarsest)	5.7E7	-	-	-	-
MLMC	8.8E7	1.2E7	5.7E6	2.7E6	1.3E6
estimator	$\ddot{N}^*_{\ell_5}$	$\ddot{N}^*_{\ell_6}$	$\ddot{N}^*_{\ell_7}$	$\ddot{N}^*_{\ell_8}$	-
MMLMC (finest)	1.2E6	-	-	2.3E5	-
MMLMC (coarsest)	-	-	-	7.8E5	-
MLMC	6.3E5	3.1E5	1.5E5	7.7E4	-

Table 2: optimal number of simulation paths, $\ddot{N}^*_{\ell_j}$: European vanilla option

- Lookback option

In comparison to the MLMC estimator, the finest MMLMC estimator, which is $\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$, achieves variance reduction of little less than 30 %. The MMLMC estimators reduce variance by 23-81 % of the SMC estimator.

estimator	estimator construction	levels	variance
MMLMC (finest)	$\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{3,0} + \tilde{Y}_{5,3} + \tilde{Y}_{8,5}$	0, 3, 5, 8	4.33E-9
MMLMC (coarsest)	$\ddot{Y} = \tilde{Y}_0 + \tilde{Y}_{8,0}$	0, 8	1.77E-8
MLMC	$\widehat{Y} = \sum_{\ell=0}^{8} \widehat{Y}_\ell$	0,1, ... , 8	6.02E-9
SMC	Y	-	2.32E-8

Table 3: comparison of variances: Lookback option

estimator	$\ddot{N}^*_{\ell_0}$	$\ddot{N}^*_{\ell_1}$	$\ddot{N}^*_{\ell_2}$	$\ddot{N}^*_{\ell_3}$	$\ddot{N}^*_{\ell_4}$
MMLMC (finest)	3.9E7	-	-	9.5E6	-
MMLMC (coarsest)	1.9E7	-	-	-	-
MLMC	3.3E7	1.2E7	6.6E6	3.5E6	1.8E6
estimator	$\ddot{N}^*_{\ell_5}$	$\ddot{N}^*_{\ell_6}$	$\ddot{N}^*_{\ell_7}$	$\ddot{N}^*_{\ell_8}$	-
MMLMC (finest)	1.8E6	-	-	3.3E5	-
MMLMC (coarsest)	-	-	-	9.2E5	-
MLMC	9E5	4.6E5	2.3E5	1.1E5	-

Table 4: optimal number of simulation paths, $\ddot{N}^*_{\ell_j}$: Lookback option

4. Concluding Remarks

The numerical results show that all the MMLMC estimators perform better than the SMC estimator. The results also show that more than 110 types of MMLMC estimators perform better than the MLMC estimator and that the finest estimator reduces variance by more than 20 % of the MLMC estimator in two European style options pricing. Thinning some of the levels in the MLMC estimator improves the estimate of variance. We conjecture that the MMLMC method is beneficial for other options valuation. Future research can have three directions. It can apply the MMLMC method to other options pricing tested in [7] or it can apply the method to American options and CVA valuation or it can estimate the optimal number of simulation paths using other methods apart from Lagrange Multipliers, such as genetic algorithms.

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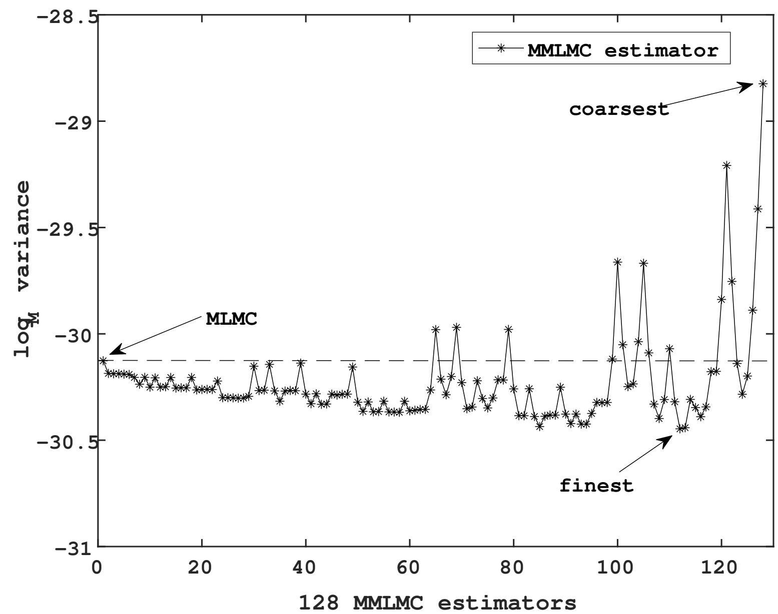

Figure 1: European vanilla option (option price: 0.10)

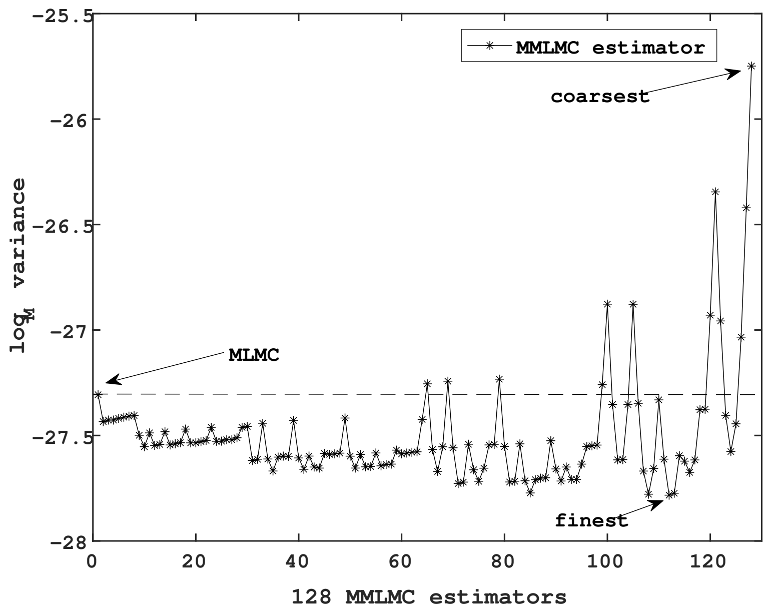

Figure 2: Lookback option (option price: 0.23)