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Abstract

Yuji Ijiri's writings on momentum accounting moved, between 1986 and 1990, from a wealth-based definition of momentum to an income-based one—a substantive shift that he did not fully articulate. We propose that this shift reflects a structural tradeoff in accounting measurement: a tension between *coherence* (the commutativity of the measurement map with the differencing operator) and *verifiability* (the support of measured quantities by independently determinable evidence). Motivated by a Hilbert-space reconstruction of Ijiri's (1986, Proposition 4) impossibility insight—that on the recirculation subspace of an accounting state space, no measurement architecture can fully satisfy both properties—we develop a functional-analytic framework that recasts the impossibility as a structural problem about institutional verification regimes. We formalize the *coherence–verifiability gap* $\varepsilon(W)$ on the recirculation subspace, derive a structural lower bound expressed in terms of the Friedrichs angle between the coherent and verifiable subspaces in operator space (Theorem 1), and construct a canonical *verifiability-extreme* measurement map $W^* = E_V \circ W^*_{\mathrm{coh}}$ whose residual coherence defect admits a closed-form expression (Theorem 2). The principal interpretive result is the closed-form expression itself: $\varepsilon_{\mathrm{coh}}(W^*) = \|[\delta_Y, E_V]W^*_{\mathrm{coh}}P_R\|_{\mathrm{HS}}$, the Hilbert–Schmidt norm of the commutator of the wealth-side differencing operator δ_Y with the verification projection E_V. The residual gap is thereby governed primarily by the structural relationship between accounting dynamics and audit-evidence partitioning, with the canonical coherent map entering as a normalized orientation; the dominant source of the residual gap is institutional rather than merely firm-specific. Applying the framework to Ijiri's 1986–1990 shift, we show that the relative ordering $\varepsilon(W_{1990})$ versus $\varepsilon(W_{1986})$ is regime-conditional: under verification regimes organized around realized income, the income-based momentum has the smaller gap; under regimes organized around balance-sheet evidence, the wealth-based momentum does. The result is not a defense of Ijiri's 1990 revision but

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a regime-conditional theory of measurement design that locates each definition on a Pareto frontier of competing demands.

Keywords: Ijiri momentum accounting; coherence; verifiability; recirculation; functional-analytic measurement theory; Friedrichs angle; conditional expectation.

JEL classification: M40, M41, M48.

1 Introduction

1.1 The 1986–1990 Momentum Puzzle

Why did Yuji Ijiri, between his 1986 paper on triple-entry bookkeeping and his 1990 monograph on *risoku* (income velocity), change the fundamental definition of accounting momentum from the time-derivative of wealth to the time-derivative of income? In *A Framework for Triple-Entry Bookkeeping* (Ijiri, 1986), Ijiri grounded the concept of momentum in stock-domain measurement: the momentum of an accounting position was the rate of change of the wealth associated with that position, with accounting flows interpreted as derivatives of stock-side state variables. Three years later, in *Momentum Accounting and Triple-Entry Bookkeeping: Exploring the Dynamic Structure of Accounting Measurements* (Ijiri, 1989), Ijiri introduced the income-based formulation as a transitional development within a primarily wealth-based architecture. By 1990, in *'Risoku Kaikei' Nyumon* (Ijiri, 1990)—a Japanese-language monograph whose title translates as "Introduction to 'Income-Velocity Accounting' "—Ijiri reformulated momentum as the time-derivative of income, treating it as a flow-domain quantity governed by accrual mechanics and the cyclical movement of items between the income statement and the balance sheet.

The change is not cosmetic. A wealth-based momentum and an income-based momentum yield distinguishable accounting frameworks: they imply different relationships between balance-sheet measurement and performance measurement, different treatments of the recirculation of items through other comprehensive income, different roles for the verification structure of audit evidence, and different correspondences with extant standards-setting practice. Yet Ijiri offered no explicit account of why he moved between the two definitions, and accounting historians have largely treated the difference as a matter of presentational convenience or evolving emphasis rather than as an artifact of underlying theoretical structure.

This paper argues that Ijiri's definitional shift reflects a fundamental, structural tradeoff in accounting measurement, and that recovering this structure illuminates not only the historical question—what changed between 1986 and 1990—but also a broader theoretical question: under what conditions does a measurement framework lie closer to, or farther from, an attainable optimum? The tradeoff in question is between two demands that any accounting measurement system places on its measurement map: *coherence*—that the rate of change of the measured state equal the measured rate of change of the state—and *verifiability*—that the measured quantities lie within the subspace supported by independently determinable evidence. We show, building on Ijiri's own structural insight, that on the recirculation subspace of an accounting state space these two demands cannot be simultaneously satisfied, and that the gap between them admits a precise functional-analytic characterization. The 1986 wealth-based definition and the 1990 income-based definition are then best understood not as competing answers to the same question, but as occupying different positions on a Pareto frontier between coherence and verifiability—with the more efficient position determined by the institutional verification regime in which the framework is embedded.

1.2 The Coherence–Verifiability Gap

The tension between coherence and verifiability is, in fact, a structural feature of any accounting measurement system that involves recirculation—that is, any system in which accounting states flow cyclically between stock-domain and flow-domain representations. Ijiri (1986) recognized this in a precise mathematical form: on the *recirculation subspace* R of an accounting state space, no correspondence map J between flow-side and stock-side measurement can simultaneously satisfy coherence and verifiability. The proof rests on the non-commutativity of the differencing operator with the measurement map—a non-commutativity that is intrinsic to the recirculation structure and that no choice of correspondence map can resolve.

Ijiri's result, however, was derived treating the measurement map W itself as given: the recirculation structure R and the verification constraints were taken as exogenous data, and the impossibility theorem was a statement about the residual relationship between the two. The natural follow-up question—what happens when W is allowed to vary, and the recirculation structure and verification constraints are treated as the data with respect to which W is designed—was not pursued. Specifically: if no measurement map achieves a zero coherence–verifiability gap, *which measurement map minimizes the gap*, how small can the minimum be, and what does the minimizer look like?

This paper takes the constructive perspective. We do not overturn Ijiri's impossibility result; we transform it. Where Ijiri's analysis treats the impossibility as a structural ceiling on what any single W can achieve, we treat it as the starting point of a structural analysis of the institutional source of the residual gap. We formalize the *coherence–verifiability gap* $\varepsilon(W)$ as a functional-analytic distance defined on the recirculation subspace R, derive a structural lower bound on $\varepsilon(W)$ expressed in terms of the Friedrichs angle between the *coherent* and *verifiable* subspaces of measurement maps in operator space, and construct a canonical *verifiability-extreme* measurement map—the unique Hilbert–Schmidt projection of a canonical coherent map onto the verifiable operator subspace—whose residual coherence defect admits a closed-form expression in terms of a single commutator. The framework thereby converts Ijiri's impossibility insight into what we will refer to throughout as a *geometric theory of accounting measurement design*: Ijiri's impossibility is not overturned but *geometrized*, with Theorem 1 giving the irreducible lower bound and Theorem 2 identifying the verifiability-extreme construction whose residual gap is governed by the commutator $[\delta_Y, E_V]$.

1.3 Contributions

The paper makes three contributions to the theory of accounting measurement.

The first contribution—the principal interpretive contribution of the paper—is the identification of the *commutator* $[\delta_Y, E_V]$ *as the structural source of the irreducible coherence–verifiability gap*. Theorem 2 constructs a canonical *verifiability-extreme* measurement map $W^* = E_V \circ W^*_{\mathrm{coh}}$, where W^*_{coh} is the canonical coherent map and E_V is the conditional expectation onto the verification σ-algebra. The construction yields zero verifiability defect by construction, and a closed-form residual coherence defect: $\varepsilon_{\mathrm{coh}}(W^*) = \|[\delta_Y, E_V]W^*_{\mathrm{coh}}P_R\|_{\mathrm{HS}}$. The two operators δ_Y and E_V entering the commutator are both structural data of the institutional environment, independent of any measurement-map choice. Ijiri (1986)'s original observation that $[\delta, W] \neq 0$

on the recirculation subspace is thereby refined: the operative non-commutativity is not between the differencing operator and the measurement map taken individually, but between the differencing operator and the verification structure that constrains the measurement map. Verifiability ceases to be a static property of available evidence and becomes a *dynamic* property of whether evidence partitions are preserved by accounting dynamics. The conflict between coherence and verifiability is, structurally, a conflict between accounting dynamics and audit-evidence partitioning, not between coherence and any specific firm-level measurement design.

The second contribution is a *structural lower bound* on the coherence–verifiability gap. Theorem 1 shows that, after imposing a non-degeneracy restriction (parametrized by c_0) to rule out trivial maps, the gap is bounded below by a spectral quantity determined by three structural ingredients: the commutator operator $L : T \mapsto \delta_Y T - T\delta_R$, the verification projection $E_V^\sharp$, and the institutional weights (α, β). The lower bound itself does not depend explicitly on c_0; the threshold defines the feasible class but does not enter the spectral expression. The bound admits a geometric form in terms of the Friedrichs angle θ_F between the coherent and verifiable subspaces in the operator space $\mathcal{L}(R, H_Y)$ when these subspaces have trivial intersection. The lower bound is *geometric* in nature: it depends not on the dimensions of the relevant subspaces but on their relative position in the operator-space metric. Two evidence structures of identical dimension can yield arbitrarily different gaps, depending on how they are positioned. The geometric framing provides a richer account of when the gap is large, when it is small, and what kinds of institutional reform would reduce it. We note that the canonical verifiability-extreme map W^* of Theorem 2 is not in general a global minimizer of the linear-weighted gap $\varepsilon = \alpha\varepsilon_{\text{coh}} + \beta\varepsilon_{\text{ver}}$; rather, it is the canonical verifiability-extreme on the Pareto frontier between coherence and verifiability defects. The relationship between the Theorem 2 residual gap value and the Theorem 1 lower bound is left as a separate question.

The third contribution is the *application* of the framework to Ijiri's 1986–1990 momentum definition shift. We work in a finite-dimensional toy model with the recirculation subspace decomposed as $R = R_A \oplus R_D \oplus R_O$, where R_A captures accrual reversals, R_D captures deferred items, and R_O captures other comprehensive income recyclables. We show that the gap functional ε admits closed-form expressions for both the wealth-based map W_{1986} and the income-based map W_{1990} in this model, and that the relative ordering $\varepsilon(W_{1990})$ versus $\varepsilon(W_{1986})$ depends on a single regime parameter ρ, defined as the alignment of the verification-supported subspace M_V with the income-side versus the stock-side dimensions of wealth. The inequality $\varepsilon(W_{1990}) < \varepsilon(W_{1986})$ holds when ρ exceeds a threshold determined by the spectral asymmetry between income-side and stock-side dynamics, and reverses when ρ falls below that threshold. The result is not a defense of Ijiri's 1990 revision but a regime-conditional characterization of when each definition occupies the more efficient position on the Pareto frontier. Under verification regimes organized around realized income, accrual reversal, and periodized performance evidence—including most of the institutional contexts in which Ijiri was working in 1990—the income-based momentum yields the smaller gap. Under verification regimes organized around balance-sheet evidence, fair-value attestation, and stock-position confirmation—including portions of the contemporary IFRS-9-influenced framework—the wealth-based momentum does.

1.4 Position Relative to Related Work

Suzuki (2026b) reconstructs Ijiri's momentum framework in noncommutative form and identifies the 1986/1990 definitional shift that motivates our application; the present paper is independent of that reconstruction in its formal apparatus and provides a quantitative analysis of the structural tradeoff between coherence and verifiability that the qualitative reconstruction does not address. Suzuki (2026a) develops a formal framework for path-dependent accounting measurement and the minimal structure of triple-entry bookkeeping; the recirculation subspace R studied here is related to but distinct from the recirculation structures analyzed there, and the present analysis can be read independently. The standards-setting endogenization of the verification structure $\mathcal{G}$—in which the regulator's choice of audit-evidence partitioning is itself the object of analysis—is beyond the scope of this paper and is left to future work.

1.5 Methodological Note and Plan of the Paper

We treat measurement maps W as bounded linear operators between Hilbert spaces, rather than as finite-dimensional matrices. This functional-analytic framing is consistent with the measurement-theoretic tradition of Mattessich (1957, 1964, 1995) and Chambers (1966), and with the operator-theoretic apparatus deployed by Christensen and Demski (2003) and Christensen and Feltham (2003). Two methodological choices warrant emphasis. First, we model verification not as a primitive list of evidence operators but as a coarsening of accounting information, represented by a sub-σ-algebra $\mathcal{G}$ of the wealth-side σ-algebra and the associated conditional expectation operator $E_V = \mathbb{E}[\,\cdot\,|\,\mathcal{G}]$. This is a substantive modeling choice, not merely a notational convenience: conditional expectation onto a sub-σ-algebra automatically preserves positivity and constants, properties that are essential for accounting admissibility but that fail for arbitrary orthogonal projections. The functional-analytic apparatus is used here as a disciplined language for making accounting measurement, verification, and recirculation jointly analyzable; it does not replace the accounting restrictions that make the construction admissible. Second, we restrict attention throughout the main text to finite-dimensional recirculation subspaces and finite-rank measurement maps. Infinite-dimensional generalizations are sketched in Appendix B.

The paper proceeds as follows. Section 2 reviews the related literature in three conversations. Section 3 develops the model. Section 4 restates Ijiri's impossibility result as a Hilbert-space reconstruction. Section 5 derives the structural lower bound on the coherence–verifiability gap (Theorem 1). Section 6 constructs the canonical verifiability-extreme map and characterizes its residual coherence defect via the commutator $[\delta_Y, E_V]$ (Theorem 2). Section 7 applies the framework to Ijiri's 1986–1990 momentum definition shift through analytic reconstruction in a finite-dimensional toy model. Section 8 concludes. Appendix A contains the proofs; Appendix B presents a numerical illustration and discusses the generalization beyond the homogeneous toy model.

2 Related Literature

2.1 Ijiri's Trilogy: 1986, 1989, 1990

Ijiri's program on triple-entry bookkeeping and momentum accounting develops across three principal works. The 1986 paper (Ijiri, 1986) introduces the recirculation concept formally, identifies the structural impossibility on the recirculation subspace (Proposition 4), and develops the wealth-based definition of momentum. The 1989 AAA monograph (Ijiri, 1989), *Momentum Accounting and Triple-Entry Bookkeeping*, is best read as a transitional work in which the wealth-based architecture is preserved overall but the income-based formulation appears in foregrounded passages—most notably in Chapter 4, Section 4.1 (p. 138), where "income momentum" is explicitly defined as the time-derivative of income. The 1990 Japanese-language monograph (Ijiri, 1990), *'Risoku Kaikei' Nyumon*, reformulates momentum as the time-derivative of income and recasts wealth as the integral of income over time—the substantive shift that motivates Section 7. The earlier *Theory of Accounting Measurement* (Ijiri, 1975) provides the broader axiomatic foundation in which the recirculation analysis is embedded. Okada and Koike (2021) surveys the development of Ijiri's program and the reception of his momentum work in the accounting research community.

The reception of Ijiri's program has been substantive but relatively narrow. The triple-entry bookkeeping framework has been engaged most extensively in the accounting-history literature and in measurement-theoretic discussions, with less direct uptake in mainstream empirical accounting research. The momentum concept, in particular, has been treated more as a theoretical curiosity than as an active research program; in part this reflects the technical depth of Ijiri's apparatus, in part the absence of clear empirical content for the recirculation subspace and the verification structure as institutional objects.

Recent work by Suzuki (2026b) reconstructs Ijiri's momentum framework in noncommutative form, formalizing the differencing operator and the measurement map as operators on a Hilbert space and the impossibility result as a non-commutativity statement on the recirculation subspace. That reconstruction also identifies the 1986/1990 inconsistency that motivates the application of Section 7. The relationship between Suzuki (2026b) and the present paper is one of complementarity rather than dependence: the former is a reconstruction of Ijiri's framework in operator-theoretic notation; the present paper takes the structural insight of the impossibility result as a starting point and develops a constructive variational framework with quantitative content that does not appear in either Ijiri's original work or its recent reconstruction. The present paper is structured to be readable independently of Suzuki (2026b).

The position of the present paper relative to Ijiri's trilogy is constructive rather than critical. We take the structural insight of Proposition 4 of Ijiri (1986)—that on the recirculation subspace, no measurement architecture can fully satisfy both coherence and verifiability—as the starting point of a variational analysis that quantifies the gap and characterizes its institutional dependencies. We do not claim to overturn Ijiri's result, refute the 1990 revision, or reinterpret Ijiri's intentions. What we add is the formal apparatus to convert Ijiri's structural insight into a quantitative theory of measurement design under verification constraints, and the analytic reconstruction that locates the 1986 and 1990 definitions on a regime-conditional Pareto frontier rather than treating them as competing universal claims.

A related but distinct connection is to Ijiri's late interdisciplinary work on quantum information and accounting information with Demski, FitzGerald, Yumi Ijiri, and Lin. Those papers explored conceptual applications of quantum information theory and topology to accounting information (Demski et al., 2006, 2009). The present paper is not an application of quantum mechanics to accounting. Its connection is more structural and more limited: it uses Hilbert-space and operator-theoretic tools—including projection, conditional expectation, and non-commutativity—to formulate a measurement problem internal to accounting. In this sense, the paper is adjacent to Ijiri's late information-theoretic program while remaining an accounting measurement paper: the projection is an audit-evidence conditional expectation, the relevant domain is the recirculation subspace, and the commutator is interpreted as the failure of evidence partitions to be preserved by accounting dynamics.

2.2 Coherence in Measurement Theory

The treatment of coherence in accounting measurement has its formal roots in the axiomatic measurement tradition exemplified by Mattessich (1957, 1964, 1995) and Chambers (1966). Mattessich's program—developed initially in Mattessich (1957) and elaborated in book form in Mattessich (1964)—treats accounting as a measurement system in the formal sense, with axioms specifying the algebraic structure of accounts, their additivity, and the properties of consolidation and aggregation; coherence enters as a structural requirement that the measurement system commute with the operations of the accounting cycle. Chambers's program, "Continuously Contemporary Accounting" (CoCoA), elevates coherence to a foundational normative criterion: a measurement system that fails to satisfy contemporary equivalence between flow-domain and stock-domain readings is rejected as inadmissible.

The coherence concept in this tradition is, formally, an algebraic commutativity requirement. The measurement map should commute with the relevant accounting operations—the differencing operator in our framework—so that the rate of change of a measured stock equals the measured rate of change of the underlying state. This commutativity is what we formalize in Section 3 as the condition $\delta_Y W = W\delta_R$ on the recirculation subspace, and what enters Section 5 as the structural ingredient encoded by the commutator operator $L : T \mapsto \delta_Y T - T\delta_R$.

The present paper differs from the Mattessich–Chambers tradition in two respects. First, we treat coherence not as a normative criterion but as a structural target whose failure is admitted when the institutional context makes joint coherence-verifiability achievement impossible. Where Chambers in particular treats violations of coherence as inadmissible measurement, we treat them as quantifiable defects whose magnitude is determined by the institutional environment. Second, we adopt a functional-analytic operator-theoretic apparatus that allows the coherence requirement to be expressed in geometric terms rather than in purely algebraic terms.

The Mattessich–Chambers tradition has had more limited engagement with the verifiability problem than with the coherence problem. The present paper's principal innovation, relative to this tradition, is to make the verifiability structure analytically tractable via the conditional expectation framework of Section 3.3, allowing coherence and verifiability to be jointly analyzed within a single Hilbert-space framework.

2.3 Verifiability and Information Structure

The information-theoretic treatment of verifiability and evidence content in accounting research has been developed most systematically in Christensen and Demski (2003), drawing on a longer tradition exemplified by Demski (1973, 1980) and connected to the broader information-economics framework of Christensen and Feltham (2003). In this tradition, verifiability is represented through the structure of available information sets, often formalized as σ-algebras or as partitions of the underlying state space, with measured quantities required to be measurable with respect to the available information σ-algebra. The framework provides the formal underpinning for treating verification as a coarsening of accounting information.

The conditional expectation operator $E_V = \mathbb{E}[\cdot \,|\, \mathcal{G}]$ that we adopt in Section 3.3 as the canonical projection onto the verification-supported subspace is a direct functional-analytic counterpart of the σ-algebra-based formulation in Christensen and Demski. Substantively, this choice of representation aligns the present paper with the information-theoretic tradition: verification is a coarsening of the wealth-side σ-algebra, and the conditional expectation is the unique orthogonal projection onto the coarsened space that preserves the measure-theoretic structure required for accounting admissibility.

The principal departure from the Christensen–Demski tradition is methodological. The present paper deploys the operator-theoretic structure of conditional expectation more aggressively than is standard in the information-content literature, exploiting the Hilbert-space structure to derive geometric and spectral results—the Friedrichs angle of Section 5.3, the commutator characterization of Section 6.3—that are difficult to access in the partition-based formulations of the Christensen–Demski tradition.

The information-content literature has generally treated the relationship between accounting dynamics and verifiability structure in static terms—what can be verified at a single point in time—rather than in dynamic terms. The present paper's emphasis on the commutator $[\delta_Y, E_V]$ between the wealth-side differencing operator and the verification projection introduces a dynamic dimension that is, to our knowledge, not previously analyzed: the irreducible coherence-verifiability gap arises not from any static incompatibility between measurement and evidence but from the failure of evidence partitions to be preserved by the dynamics of wealth.

3 Model

3.1 Accounting State Space and the Differencing Operator

We model accounting states as elements of a probability space. Let $(\mathcal{Z}, \mathcal{F}_Z, \mu)$ be a *probability space* ($\mu(\mathcal{Z}) = 1$) where each $z \in \mathcal{Z}$ encodes a complete configuration of accounting positions at one instant of the firm's recurring measurement cycle. Let $H_Z := L^2(\mathcal{Z}, \mathcal{F}_Z, \mu)$ denote the Hilbert space of square-integrable real-valued functions on $\mathcal{Z}$. An *accounting reading* is an element $f \in H_Z$, identified up to μ-null sets. Note that the probability normalization ensures $\mathbf{1}_{\mathcal{Z}} \in H_Z$, which is required for the constant-preservation condition (W4) in Definition 3.3 below to be well-formulated.

The *differencing operator* on the state space is a bounded linear operator $\delta_Z : H_Z \to H_Z$

with operator norm $\|\delta_Z\|_{\mathrm{op}} \leq K_Z < \infty$. Its accounting interpretation is the period-to-period change of any accounting reading. In parallel, we introduce a wealth/income reading probability space $(\mathcal{Y}, \mathcal{F}_Y, \nu)$ ($\nu(\mathcal{Y}) = 1$) with associated Hilbert space $H_Y := L^2(\mathcal{Y}, \mathcal{F}_Y, \nu)$ and a bounded *wealth-side differencing operator* $\delta_Y : H_Y \to H_Y$, $\|\delta_Y\|_{\mathrm{op}} \leq K_Y < \infty$.

Notational discipline. We emphasize that δ_Z and δ_Y are *distinct operators on distinct spaces*. This separation is essential for the commutator construction in Section 3.6 to be well-defined.

3.2 The Recirculation Subspace

The *recirculation subspace* $R \subseteq H_Z$ is the closed linear subspace of accounting readings whose values are involved in the cyclical movement between flow-domain and stock-domain measurement. Substantively, R contains those readings affected by phenomena such as accrual reversals, deferred revenue and expense items, OCI recyclables, and inventory cost flows.

Assumption 3.1 (Finite-dimensional recirculation)**.** *R is a closed finite-dimensional subspace of H_Z with $\dim(R) = r < \infty$.*

Assumption 3.2 (δ_Z-invariance of R)**.** *$\delta_Z(R) \subseteq R$.*

The interpretation of Assumption 3.2 is direct: a reading whose value lies in the recirculation subspace, when differenced, yields another reading still in the recirculation subspace. The assumption defines a restriction $\delta_R := \delta_Z|_R : R \to R$, a bounded operator on the finite-dimensional space R.

For the application of Section 7, we use the explicit decomposition $R = R_A \oplus R_D \oplus R_O$, where R_A captures accrual reversals, R_D deferred items, and R_O OCI-type recyclables. Let $P_R : H_Z \to R$ denote the orthogonal projection onto R.

3.3 Verification as Information Coarsening

Definition 3.1 (Verification σ-algebra)**.** A *verification σ-algebra* is a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}_Y$ of the wealth-side σ-algebra. A set $A \in \mathcal{G}$ corresponds to an event whose occurrence can be independently verified through audit evidence, third-party confirmation, contractual documentation, or other means recognized by the institutional framework.

Definition 3.2 (Verification-supported measurement subspace)**.** The *verification-supported measurement subspace* associated with $\mathcal{G}$ is

$$M_V := L^2(\mathcal{Y}, \mathcal{G}, \nu|_{\mathcal{G}}) \hookrightarrow H_Y,$$

identified with its image in H_Y via the natural embedding. M_V is a closed linear subspace of H_Y.

The canonical orthogonal projection $H_Y \to M_V$ is the conditional expectation $E_V := \mathbb{E}[\,\cdot \mid \mathcal{G}]$. The operator E_V has the standard properties: it is bounded with $\|E_V\|_{\mathrm{op}} = 1$; idempotent ($E_V^2 = E_V$); the orthogonal projection onto M_V in H_Y; preserves positivity ($f \geq 0 \Rightarrow E_V f \geq$

0); and preserves constants ($E_V[\mathbf{1}_Y] = \mathbf{1}_Y$). The last two properties distinguish conditional expectation from arbitrary orthogonal projections, and they are precisely what is required for the admissibility of a measurement map.

Accounting interpretation. The σ-algebra $\mathcal{G}$ captures the "audit-evidence partition"—the granularity at which independent verification can distinguish between accounting states.

3.4 Measurement Maps and the Admissible Class

A *measurement map* is a bounded linear operator $W : H_Z \to H_Y$.

Definition 3.3 (Admissibility)**.** A measurement map W is *admissible* if it satisfies: (W1) linearity; (W2) boundedness; (W3) positivity preservation ($f \geq 0 \Rightarrow Wf \geq 0$); (W4) constant preservation ($W[\mathbf{1}_Z] = \mathbf{1}_Y$). Denote the class of admissible measurement maps by $\mathcal{W}$.

3.5 Non-Degeneracy Normalization

We must restrict to a non-degenerate subclass of $\mathcal{W}$ to prevent trivial solutions to the gap-minimization problem.

Definition 3.4 (Non-degenerate admissible class)**.** For fixed $c_0 > 0$, the *non-degenerate admissible class* is

$$\mathcal{W}_0 := \{W \in \mathcal{W} : \|WP_R\|_{\mathrm{HS}} = 1,\ \sigma_{\min}(W|_R) \geq c_0\},$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm and $\sigma_{\min}$ is the smallest singular value.

Feasibility constraint on c_0. The two conditions defining $\mathcal{W}_0$ impose a feasibility constraint on c_0. Writing the singular values of $W|_R$ as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, the unit Hilbert–Schmidt norm condition gives $\sum_{i=1}^r \sigma_i^2 = 1$, while the smallest-singular-value condition requires $\sigma_r \geq c_0$. Combining the two,

$$1 = \sum_{i=1}^{r} \sigma_i^2 \geq r\, c_0^2,$$

which forces $c_0 \leq 1/\sqrt{r}$. We therefore assume throughout

$$0 < c_0 \leq r^{-1/2}, \tag{1}$$

and that $\mathcal{W}_0 \neq \emptyset$. The latter is an additional admissible-extension assumption: not every operator $T : R \to H_Y$ satisfying the singular-value constraints need extend to a positive unital measurement map on H_Z, and we require throughout that at least one such admissible extension exists. (In the boundary case $c_0 = 1/\sqrt{r}$, the singular-value constraints force $\sigma_i = 1/\sqrt{r}$ for all i.) Substantively, the feasibility constraint says that the institutional non-degeneracy threshold cannot exceed the equal-distribution lower bound at unit total measurement scale.

Accounting interpretation. The condition $\|WP_R\|_{\mathrm{HS}} = 1$ fixes the overall scale of measurement on R. The condition $\sigma_{\min}(W|_R) \geq c_0$ is substantive: it requires admissible measurements to preserve a minimum amount of recirculation information in *every* direction of R.

3.6 The Coherence–Verifiability Gap

Definition 3.5 (Coherence–verifiability gap)**.** For any bounded measurement map $W : H_Z \to H_Y$, define:

$$\varepsilon_{\mathrm{coh}}(W) := \|(\delta_Y W - W\delta_Z)P_R\|_{\mathrm{HS}} \quad \text{(coherence defect)},$$
$$\varepsilon_{\mathrm{ver}}(W) := \|(I_Y - E_V)WP_R\|_{\mathrm{HS}} \quad \text{(verifiability defect)}.$$

Since R is finite-dimensional and P_R is a finite-rank projection, the operators $(\delta_Y W - W\delta_Z)P_R$ and $(I_Y - E_V)WP_R$ are finite-rank, and these Hilbert–Schmidt norms are finite for every bounded W. The *coherence–verifiability gap* is

$$\boxed{\varepsilon(W) := \alpha\, \varepsilon_{\mathrm{coh}}(W) + \beta\, \varepsilon_{\mathrm{ver}}(W),} \tag{2}$$

where $\alpha, \beta > 0$ are institutional weights. The variational problems of Sections 5–6 restrict these defect functionals to the admissible classes $\mathcal{W}_0$ or $\mathcal{W}_0^f$ (introduced in Section 5.1).

Accounting interpretation. $\varepsilon_{\mathrm{coh}}(W)$ measures, on the recirculation subspace, how far the measurement map fails to commute with differencing; the composition $\delta_Y W$ first measures and then takes the period change, while $W\delta_Z$ first takes the state change and then measures. Coherence demands $\delta_Y W = W\delta_Z$ on R. $\varepsilon_{\mathrm{ver}}(W)$ measures the part of the measured recirculation reading that lies outside the verification-supported subspace. The weights α, β encode the institutional tradeoff between coherence-driven and verification-driven concerns. The defects are well-defined for all bounded W (not only for $W \in \mathcal{W}_0$); this broader domain matters in Section 6, where Theorem 2 constructs $W^* = E_V \circ W^*_{\mathrm{coh}} \in \mathcal{W}$ and evaluates $\varepsilon_{\mathrm{coh}}(W^*)$, $\varepsilon_{\mathrm{ver}}(W^*)$ without requiring $W^* \in \mathcal{W}_0$.

3.7 The Coherent and Verifiable Subspaces in Operator Space

The variational analysis of Sections 5 and 6 is conducted within the operator space $\mathcal{L}(R, H_Y)$. Since R is finite-dimensional, $\mathcal{L}(R, H_Y)$ is a Hilbert space under the Hilbert–Schmidt inner product

$$\langle S, T\rangle_{\mathrm{HS}} := \sum_{i=1}^{r} \langle Se_i, Te_i\rangle_{H_Y},$$

where $\{e_i\}_{i=1}^r$ is any orthonormal basis of R.

Definition 3.6 (Coherent subspace)**.** The *coherent subspace* is

$$\mathcal{C} := \{T \in \mathcal{L}(R, H_Y) : \delta_Y T - T\delta_R = 0\},$$

equivalently the kernel of the bounded linear *commutator operator* $L : T \mapsto \delta_Y T - T\delta_R$.

Definition 3.7 (Verifiable subspace)**.** The *verifiable subspace* is $\mathcal{V} := \mathcal{L}(R, M_V) = \{T \in \mathcal{L}(R, H_Y) : \mathrm{Range}(T) \subseteq M_V\}$.

3.8 The Central Question

The paper's central question is:

> **(Q)** What is $\inf_{W \in \mathcal{W}_0} \varepsilon(W)$, and which W^* attains the infimum, if attainable?

The negative answer to "is there $W \in \mathcal{W}_0$ with $\varepsilon(W) = 0$?" is essentially Ijiri's impossibility result, restated formally in Section 4 (Lemma 1). Section 5 (Theorem 1) gives a structural *lower bound* for the relaxed finite-dimensional operator-space problem and, through the inclusion $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$, for the finite-rank admissible problem $\mathcal{W}_0^f$ introduced below. Without the finite-rank restriction $\mathcal{W}_0 = \mathcal{W}_0^f$, no general ordering is asserted between the relaxed lower bound and the full admissible infimum over $\mathcal{W}_0$. Section 6 (Theorem 2) does *not* solve the full weighted minimization in (Q); rather, it constructs a canonical *verifiability-extreme* measurement map—the unique HS-projection of a selected coherent representative onto the verifiable operator subspace—and characterizes its residual coherence defect via the commutator $[\delta_Y, E_V]$. The relationship between the Theorem 1 lower bound and the Theorem 2 verifiability-extreme defect is left as a separate question.

4 Impossibility on the Recirculation Subspace

This section restates Ijiri's impossibility insight in the model developed in Section 3 and clarifies the boundaries of the result. Section 4.1 reviews Ijiri's original formulation. Section 4.2 develops the *Hilbert-space reconstruction* of Ijiri's insight, formalized as Lemma 1. Section 4.3 traces the impossibility to a non-commutativity sharper than the one Ijiri identified: not between the differencing operator and the measurement map, but between the differencing operator and the verification projection. Section 4.4 articulates what the result leaves open.

4.1 Ijiri's Original Insight

Ijiri's analysis of momentum accounting (Ijiri, 1986, 1989, 1990) was built on the observation that a recurrent accounting cycle imposes a structural relationship between flow-domain and stock-domain measurement. In the cycle, accounting positions move repeatedly between the income statement and the balance sheet—accruals reverse, deferred items realize, OCI items recycle through net income—and the measurement map is required to produce mutually consistent readings on both sides of the cycle.

In Ijiri (1986), this consistency requirement was articulated through a correspondence map between flow-side and stock-side measurements, and through two structural demands placed on it: that the rate of change of a measured stock equal the measured rate of change of the underlying state ("coherence"), and that the measured quantities be supported by independently determinable evidence ("verifiability"). Ijiri's Proposition 4 established that, on the recirculation subspace of the state space, no correspondence map satisfying standard regularity conditions could simultaneously meet both demands.

We do not restate Ijiri's theorem verbatim. Ijiri's notation, conventions, and ambient framework differ in small but consequential ways from those of the present paper, and a verbatim restatement would either misrepresent Ijiri's text or require a layer of translation that obscures the structural content. Instead, the next subsection presents the *Hilbert-space reconstruction* of Ijiri's impossibility insight: a formal statement, in the model of Section 3, that captures the essential structural content of his argument.

4.2 Hilbert-Space Reconstruction

We restate Ijiri's insight in the notation of the present paper. The translation is structural rather than textual: we take Ijiri's recirculation subspace, his coherence and verifiability conditions, and his impossibility conclusion, and re-express them in the framework of Section 3.

Preview of Section 5 notation. For the formal statement of the lemma below, we use four objects that will be defined fully in Section 5.1 (Assumption 5.1 and the subsequent paragraph). For the present, the reader may take the following as a working preview: (i) $\mathcal{H}_{\mathrm{op}} := \mathcal{L}(R, M_f)$, the finite-dimensional working operator space, where $M_f \subseteq H_Y$ is a finite-dimensional subspace containing the verification-supported subspace M_V and the ranges of the relevant admissible measurement maps; (ii) $\mathcal{C}_f := \mathcal{C} \cap \mathcal{H}_{\mathrm{op}}$, the coherent subspace within $\mathcal{H}_{\mathrm{op}}$; (iii) $\mathcal{V}_f := \mathcal{L}(R, M_V) \subseteq \mathcal{H}_{\mathrm{op}}$, the verifiable subspace; and (iv) $\mathcal{T}_0 := \{T \in \mathcal{H}_{\mathrm{op}} : \|T\|_{\mathrm{HS}} = 1,\ \sigma_{\min}(T) \geq c_0\}$, the relaxed operator-space class corresponding to $\mathcal{W}_0$.

Lemma 1 (Hilbert-space reconstruction of Ijiri's impossibility)**.** *Under Assumptions 3.1 and 3.2, and the finite-rank reduction of Assumption 5.1 (Section 5), the relaxed gap*

$$\varepsilon^*_{\mathcal{T}_0} := \inf_{T \in \mathcal{T}_0} \left[\alpha \|L_f T\|_{\mathrm{HS}} + \beta \|(I_{\mathcal{H}_{\mathrm{op}}} - E_V^\sharp) T\|_{\mathrm{HS}}\right].$$

is attained on $\mathcal{T}_0$. Moreover,

$$\varepsilon^*_{\mathcal{T}_0} > 0 \quad \Longleftrightarrow \quad \mathcal{C}_f \cap \mathcal{V}_f \cap \{T \in \mathcal{H}_{\mathrm{op}} : \sigma_{\min}(T) \geq c_0,\ \|T\|_{\mathrm{HS}} = 1\} = \emptyset.$$

Since $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$, the relaxed gap is a lower bound on the finite-rank admissible gap: $\varepsilon^*_{\mathcal{T}_0} \leq \varepsilon^*_{\mathcal{W},f}$. Since $\mathcal{W}_0^f \subseteq \mathcal{W}_0$, one also has $\varepsilon^*_{\mathcal{W}} \leq \varepsilon^*_{\mathcal{W},f}$. No general ordering between $\varepsilon^*_{\mathcal{T}_0}$ and $\varepsilon^*_{\mathcal{W}}$ is asserted without the additional restriction $\mathcal{W}_0 = \mathcal{W}_0^f$.

The right-hand side reads: in the relaxed operator-space problem, there is no element of $\mathcal{T}_0$ that is simultaneously coherent and verifiable on R. Lemma 1 thereby converts Ijiri's impossibility insight into a precise statement about the (non-)existence of operator-space candidates satisfying both structural demands. The corresponding statement for the finite-rank admissible problem follows: if no element of $\mathcal{T}_0$ achieves zero gap, *a fortiori* no element of $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$ does, and hence $\varepsilon^*_{\mathcal{W},f} > 0$.

The relationship between Lemma 1 and Proposition 4 of Ijiri (1986) is one of structural correspondence rather than verbatim translation. Ijiri formulated the impossibility as a property of the correspondence map between flow-side and stock-side measurements, with the measurement architecture taken as fixed. Our reconstruction holds the recirculation structure R, the verification σ-algebra $\mathcal{G}$, and the ambient differencing operators δ_Z, δ_Y as fixed, and quantifies over the admissible class $\mathcal{W}_0^f$. The two formulations capture the same underlying tension—between the differencing of measured states and the joint coherence/verifiability of the resulting measurements—but distribute the variable and the data differently and operate at different levels of generality. The reconstruction has two virtues for our purposes: it places the impossibility within a variational framework that admits the constructive analysis of Sections 5 and 6, and it makes the structural source of the impossibility—to which we now turn—formally explicit. The proof of Lemma 1 is given in Appendix A.1.

4.3 The Underlying Non-Commutativity

Ijiri (1986) located the source of the impossibility in the failure of the differencing operator to commute with the measurement map: $[\delta, W] \neq 0$ on the recirculation subspace. In our framework, this commutator structure can be unpacked into a more fundamental non-commutativity that does not depend on the choice of W.

Suppose, for the sake of argument, that there exists $W \in \mathcal{W}_0$ that is simultaneously coherent and verifiable on R. For any $f \in R$:

- by verifiability, $Wf \in M_V$;
- by coherence, $\delta_Y W f = W \delta_R f$;
- by δ_Z-invariance of R (Assumption 3.2), $\delta_R f \in R$;
- by verifiability applied to $\delta_R f$, $W\delta_R f \in M_V$.

Combining,

$$\delta_Y W f = W\delta_R f \in M_V \quad \text{for every } f \in R.$$

The range $W(R) \subseteq M_V$ would therefore have to lie in the *one-step-δ_Y-preserving subspace* of M_V—the set of vectors in M_V whose image under δ_Y remains in M_V.

The size and structure of the one-step-preserving subspace are determined entirely by the relationship between δ_Y and E_V, independently of W. The relevant condition for M_V to be δ_Y-invariant is the vanishing of the *leakage component* $(I - E_V)\delta_Y E_V$: when this leakage is zero, the verification-dynamics obstruction disappears—δ_Y does not force verifiable measurements out of M_V—though joint coherence-verifiability may still require a non-degenerate intertwiner between δ_R and δ_Y and an admissible positive unital extension. When the leakage component is non-zero, the one-step-preserving subspaces are strictly smaller than M_V, and joint coherence-verifiability becomes correspondingly harder to achieve. In the limit, when no δ_Y-invariant subspace of M_V has dimension at least $r = \dim R$, the constraint $W(R) \subseteq$ (one-step-preserving subspace) cannot be satisfied by any non-degenerate W, and $\varepsilon^*_{\mathcal{W},f} > 0$ follows.

This observation refines Ijiri's original framing in a substantive way. Where Ijiri attributed the impossibility to the non-commutativity of the differencing operator with the measurement map—a property of the chosen W—the refined attribution is to the non-commutativity of the wealth-side differencing operator with the verification projection: $[\delta_Y, E_V] \neq 0$, which decomposes as

$$[\delta_Y, E_V] = (I - E_V)\,\delta_Y\,E_V \;-\; E_V\,\delta_Y\,(I - E_V).$$

The first summand is the *leakage component* $(I - E_V)\delta_Y E_V$, which governs the obstruction to verifiable coherence as identified above (whether M_V is δ_Y-invariant). The second summand $E_V\delta_Y(I-E_V)$ captures the converse *mixing* from $M_V^\perp$ into M_V. Theorem 2 will package the residual coherence defect of the canonical verifiability-extreme map through the full commutator $[\delta_Y, E_V]$, thereby incorporating both effects into a single quantitative measure. Both components are structural properties of the institutional verification regime, independent of any particular measurement map. The coherence-verifiability gap, in this reading, is not a property of how the firm measures but of how the dynamics of wealth interact with the partitioning of audit-relevant evidence. The conflict is between accounting time and audit information, prior to any choice the firm makes about how to map the former into the latter.

We record this as a working observation. The quantitative version, in which the coherence defect of the canonical verifiability-extreme map W^* is shown to equal $\|[\delta_Y, E_V]W^*_{\mathrm{coh}}P_R\|_{\mathrm{HS}}$, appears as part of Theorem 2 in Section 6.

4.4 The Boundary of the Impossibility

Lemma 1 establishes attainment and strict positivity for the relaxed operator-space problem on $\mathcal{T}_0$, and yields a sufficient lower-bound statement for the finite-rank admissible problem because $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$. Three observations about what Lemma 1 does *not* directly establish are useful for situating the constructive analysis that follows.

First, Lemma 1 does not quantify the relaxed gap. It establishes that $\varepsilon^*_{\mathcal{T}_0} > 0$ (under the impossibility condition) and *a fortiori* $\varepsilon^*_{\mathcal{W},f} \geq \varepsilon^*_{\mathcal{T}_0} > 0$, but does not specify how large $\varepsilon^*_{\mathcal{T}_0}$ is, what structural ingredients determine its magnitude, or how that magnitude varies with changes in the recirculation subspace, the verification σ-algebra, or the ambient differencing operators. Section 5 fills this gap by deriving a structural lower bound on $\varepsilon^*_{\mathcal{T}_0}$ in terms of the Friedrichs angle between $\mathcal{C}_f$ and $\mathcal{V}_f$ and the spectral data of the commutator operator $L_f : T \mapsto \delta_Y T - T\delta_R$.

Second, Lemma 1 does not identify the measurement map (if any) that minimizes the gap on the full admissible class $\mathcal{W}_0$. Lemma 1 establishes attainment of the relaxed infimum on $\mathcal{T}_0$, not of the full admissible infimum on $\mathcal{W}_0$; the latter would require additional closedness or compactness assumptions on $\mathcal{T}_{\mathcal{W}}$. Section 6 does not solve the full weighted minimization problem; instead, it constructs a canonical *verifiability-extreme* measurement map $W^* = E_V \circ W^*_{\mathrm{coh}}$—the unique HS-orthogonal projection of a canonical coherent map onto the verifiable operator subspace—and derives a closed-form expression for its residual coherence defect.

Third, Lemma 1 treats both the recirculation structure R and the verification σ-algebra $\mathcal{G}$ as exogenous. In the institutional reality of accounting, both are themselves the products of design choices. A complete theory of measurement design would endogenize both, treating them as choice variables of a regulator or a standards-setter. Such endogenization is beyond the scope of the present paper.

The three observations together identify the question to which the remainder of the paper is addressed: given $(R, \mathcal{G}, \delta_Z, \delta_Y)$, how large is the coherence-verifiability gap, what is its structural source, and which admissible measurement map sits at the verifiability-extreme of the Pareto frontier?

5 The Coherence–Verifiability Lower Bound

This section quantifies the impossibility by deriving a structural lower bound on the gap functional ε that depends on the geometric relationship between the coherent and verifiable subspaces in operator space. The result, Theorem 1, takes a spectral form: the lower bound is the smallest eigenvalue of a specific positive self-adjoint operator A_f on the finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{op}} = \mathcal{L}(R, M_f)$ (introduced via Assumption 5.1 below). This spectral form admits a geometric interpretation in terms of the *Friedrichs angle* (Friedrichs, 1937) between the coherent subspace $\mathcal{C}_f$ and the verifiable subspace $\mathcal{V}_f$ within $\mathcal{H}_{\mathrm{op}}$.

5.1 The Gap as a Variational Problem on Operator Space

We begin by recasting the gap minimization problem in a form that exposes its variational structure. The gap functional defined in (2) depends on W only through the composition $WP_R : H_Z \to H_Y$. Since P_R has range R, this composition factors as $WP_R = (W|_R) \circ P_R$, and the defects depend on W only through the restriction $W|_R$.

Recall the commutator operator $L : T \mapsto \delta_Y T - T\delta_R$ from Definition 3.6. A direct estimate using the standard Hilbert–Schmidt inequalities yields $\|LT\|_{\mathrm{HS}} \le (\|\delta_Y\|_{\mathrm{op}} + \|\delta_R\|_{\mathrm{op}})\|T\|_{\mathrm{HS}}$, so L is bounded with $\|L\|_{\mathrm{op}} \le \|\delta_Y\|_{\mathrm{op}} + \|\delta_R\|_{\mathrm{op}}$. By construction, $\ker L = \mathcal{C}$. Under Assumption 5.1 below, we work with the restriction $L_f := L|_{\mathcal{H}_{\mathrm{op}}} : \mathcal{H}_{\mathrm{op}} \to \mathcal{H}_{\mathrm{op}}$, with $\ker L_f = \mathcal{C}_f$. The coherence defect of any $W \in \mathcal{W}_0$ is the Hilbert–Schmidt norm of L_f applied to WP_R: $\varepsilon_{\mathrm{coh}}(W) = \|L_f(WP_R)\|_{\mathrm{HS}}$.

The second key operator is the *operator-space verification projection* $E_V^\sharp : \mathcal{L}(R, H_Y) \to \mathcal{L}(R, H_Y)$ defined by $E_V^\sharp T := E_V \circ T$.

Lemma 2. *$E_V^\sharp$ is the orthogonal projection from $\mathcal{L}(R, H_Y)$ onto $\mathcal{V}$ in the Hilbert–Schmidt structure. Under Assumption 5.1, $E_V^\sharp$ restricts to the orthogonal projection from $\mathcal{H}_{\mathrm{op}}$ onto $\mathcal{V}_f$.*

The proof of Lemma 2 is given in Appendix A.2. The verifiability defect is then $\varepsilon_{\mathrm{ver}}(W) = \|(I_{\mathcal{L}} - E_V^\sharp)(WP_R)\|_{\mathrm{HS}}$, where $I_{\mathcal{L}}$ denotes the identity on $\mathcal{L}(R, H_Y)$ (which restricts to $I_{\mathcal{H}_{\mathrm{op}}}$ on $\mathcal{H}_{\mathrm{op}}$). The coherent and verifiable subspaces admit two equivalent characterizations as bounded-linear-operator-defined subsets of $\mathcal{L}(R, H_Y)$: $\mathcal{C} = \ker L$, and $\mathcal{V} = \mathrm{Range}(E_V^\sharp) = \ker(I_{\mathcal{L}} - E_V^\sharp)$, with finite-dimensional restrictions $\mathcal{C}_f := \mathcal{C} \cap \mathcal{H}_{\mathrm{op}}$ and $\mathcal{V}_f := \mathcal{V}$ (since $\mathcal{V} = \mathcal{L}(R, M_V) \subseteq \mathcal{L}(R, M_f) = \mathcal{H}_{\mathrm{op}}$).

We adopt the additional finite-rank reduction.

Assumption 5.1 (Finite-dimensional working operator space)**.** There exists a finite-dimensional closed subspace $M_f \subseteq H_Y$ such that:

- the verification-supported subspace satisfies $M_V \subseteq M_f$;
- $\delta_Y(M_f) \subseteq M_f$ (so δ_Y restricts to a bounded operator on M_f).

We work throughout with the finite-dimensional operator space

$$\mathcal{H}_{\mathrm{op}} := \mathcal{L}(R, M_f)$$

endowed with the Hilbert–Schmidt inner product, on which all the operator-theoretic constructions of this section (L, $E_V^\sharp$, A) act as bounded linear maps with finite-dimensional domain and range. We further introduce the *finite-rank admissible class*

$$\mathcal{W}_0^f := \{W \in \mathcal{W}_0 : \mathrm{Range}(WP_R) \subseteq M_f\},$$

and assume $\mathcal{W}_0^f \neq \emptyset$. Throughout Sections 5–6, the variational analysis is conducted over $\mathcal{W}_0^f$. We denote the three corresponding infima as

$$\begin{aligned}
\varepsilon^*_{\mathcal{W}} &:= \inf_{W\in\mathcal{W}_0} \varepsilon(W) \qquad \text{(full admissible gap)},\\
\varepsilon^*_{\mathcal{W},f} &:= \inf_{W\in\mathcal{W}_0^f} \varepsilon(W) \qquad \text{(finite-rank admissible gap)},\\
\varepsilon^*_{\mathcal{T}_0} &:= \inf_{T\in\mathcal{T}_0} \left[\alpha\|L_f T\|_{\mathrm{HS}} + \beta\|(I_{\mathcal{H}_{\mathrm{op}}} - E_V^\sharp)T\|_{\mathrm{HS}}\right] \qquad \text{(relaxed operator-space gap)}.
\end{aligned}$$

The relationships are $\varepsilon^*_{\mathcal{W}} \leq \varepsilon^*_{\mathcal{W},f}$ (since $\mathcal{W}_0^f \subseteq \mathcal{W}_0$) and $\varepsilon^*_{\mathcal{T}_0} \leq \varepsilon^*_{\mathcal{W},f}$ (since $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$). The principal lower bound of Section 5 (Theorem 1) is on $\varepsilon^*_{\mathcal{T}_0}$ and propagates to $\varepsilon^*_{\mathcal{W},f}$ via the second inclusion.

Finite-dimensional working objects. Henceforth, we write the operator-theoretic objects of Sections 5–6 in their finite-dimensional restrictions on $\mathcal{H}_{\mathrm{op}}$:

$$\begin{aligned}
L_f : \mathcal{H}_{\mathrm{op}} \to \mathcal{H}_{\mathrm{op}}, \quad L_f T &:= \delta_Y T - T\delta_R;\\
\mathcal{C}_f &:= \ker(L_f) \subseteq \mathcal{H}_{\mathrm{op}} \quad \text{(coherent subspace)};\\
\mathcal{V}_f &:= \mathcal{L}(R, M_V) \subseteq \mathcal{H}_{\mathrm{op}} \quad \text{(verifiable subspace, since } M_V \subseteq M_f\text{)};\\
E_V^\sharp : \mathcal{H}_{\mathrm{op}} \to \mathcal{H}_{\mathrm{op}}, \quad E_V^\sharp T &:= E_V \circ T \quad \text{(HS-orthogonal projection onto } \mathcal{V}_f\text{)};\\
A_f &:= \alpha^2 L_f^* L_f + \beta^2 (I_{\mathcal{H}_{\mathrm{op}}} - E_V^\sharp).
\end{aligned}$$

L_f, $E_V^\sharp$, and A_f are bounded linear operators on the finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{op}}$. Consequently, A_f has a discrete spectrum and the smallest eigenvalue $\lambda_{\min}(A_f)$ is well-defined as the minimum of the (finite) spectrum.

The institutional content of the finite-rank reduction is that, in any single accounting period, only finitely many distinguishable evidence categories support verification, and only finitely many wealth-side directions are relevant to the recirculation analysis. The infinite-dimensional case, in which $\lambda_{\min}(A_f)$ may need to be replaced by the bottom of the essential spectrum, is sketched in Appendix B.3.

The variational problem $\inf_{W\in\mathcal{W}_0^f} \varepsilon(W)$ thereby translates into a constrained minimization over $\mathcal{H}_{\mathrm{op}}$. The natural operator-space variable is the restriction $T := W|_R \in \mathcal{H}_{\mathrm{op}}$ (an operator $R \to M_f$); the related composition $WP_R : H_Z \to M_f$ acts as $WP_R = T \circ P_R$ and shares the Hilbert–Schmidt norm $\|WP_R\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$. With this convention, we distinguish two operator-space classes:

$$\mathcal{T}_{\mathcal{W}} := \{T \in \mathcal{H}_{\mathrm{op}} : T = W|_R \text{ for some } W \in \mathcal{W}_0^f\} \qquad \text{(image of } \mathcal{W}_0^f\text{, “full admissible”),}$$
$$\mathcal{T}_0 := \{T \in \mathcal{H}_{\mathrm{op}} : \|T\|_{\mathrm{HS}} = 1,\ \sigma_{\min}(T) \geq c_0\} \qquad \text{(“relaxed operator-space”).}$$

We have $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$ but the inclusion can be strict: a generic $T \in \mathcal{T}_0$ need not extend to a positive unital map $W : H_Z \to H_Y$ on the full state space, so the operator-space norm and singular-value constraints do not imply membership in $\mathcal{W}_0^f$. Correspondingly, the gap minimization decomposes as

$$\sqrt{\lambda_{\min}(A_f)} \ \leq\ \varepsilon^*_{\mathcal{T}_0} \ \leq\ \varepsilon^*_{\mathcal{W},f},$$

where the leftmost inequality is the spectral lower bound established by Theorem 1 below (proven on the relaxed class $\mathcal{T}_0 \subseteq \mathcal{H}_{\mathrm{op}}$, hence valid *a fortiori* on the smaller class $\mathcal{T}_{\mathcal{W}}$). Throughout Sections 5 and 6 we work primarily with the relaxed class $\mathcal{T}_0$, which is a closed bounded subset of the finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{op}}$ and hence compact; the relationship to the finite-rank admissible problem $\mathcal{W}_0^f$ is mediated by the positive-unital-extension assumption made explicit in Assumption 6.1 below.

5.2 Theorem 1: Spectral Lower Bound

Theorem 1 (Structural lower bound on the coherence–verifiability gap)**.** *Under Assumptions 3.1, 3.2, and 5.1, the relaxed gap satisfies*

$$\varepsilon^*_{\mathcal{T}_0} \geq \sqrt{\lambda_{\min}(A_f)},$$

where $A_f := \alpha^2 L_f^* L_f + \beta^2 (I_{\mathcal{H}_{\mathrm{op}}} - E_V^\sharp)$ is the bounded self-adjoint positive semi-definite operator on the finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{op}} = \mathcal{L}(R, M_f)$ defined above, and $\lambda_{\min}(A_f)$ denotes its smallest eigenvalue. Since $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$, the bound holds a fortiori *for the finite-rank admissible problem: $\varepsilon^*_{\mathcal{W},f} \geq \varepsilon^*_{\mathcal{T}_0} \geq \sqrt{\lambda_{\min}(A_f)}$. Moreover, $\lambda_{\min}(A_f) > 0$ if and only if $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$.*

The proof is given in Appendix A.3.

Remark 5.1 (Proportional scaling)*.* The lower bound scales linearly in the institutional weights: replacing (α, β) by $(\lambda\alpha, \lambda\beta)$ for $\lambda > 0$ scales A_f by λ^2 and hence $\sqrt{\lambda_{\min}(A_f)}$ by λ.

Remark 5.2 (Structural data dependencies)*.* The lower bound is determined by exactly four pieces of structural data: the differencing operators δ_Y and δ_R (entering through L_f); the verification σ-algebra $\mathcal{G}$ (entering through $E_V^\sharp$); the institutional weights α, β; and the finite-dimensional ambient operator space $\mathcal{H}_{\mathrm{op}} = \mathcal{L}(R, M_f)$. The bound is independent of any specific measurement map. No finite-rank non-degenerate admissible measurement design in $\mathcal{W}_0^f$ can yield a gap below $\sqrt{\lambda_{\min}(A_f)}$; correspondingly, the broader rhetorical claim that “no measurement design can do better” is to be understood within the scope of the present framework, on the recirculation subspace R, with verification structure $\mathcal{G}$, and within the admissible class $\mathcal{W}_0^f$.

Remark 5.3 (The non-degeneracy threshold c_0)*.* The lower bound $\sqrt{\lambda_{\min}(A_f)}$ does not involve c_0 explicitly. The threshold is required to ensure that $\mathcal{T}_0$ excludes degenerate (rank-deficient) elements that would render the gap problem trivial, but the spectral expression itself is determined by A_f alone. A sharper analysis exploiting c_0 can yield improved lower bounds in regimes where this constraint binds, but is not pursued here.

Remark 5.4 (Comparison with Lemma 1's positivity condition). The condition $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$ is the condition under which the displayed spectral lower bound is strictly positive. It is logically stronger than the non-degenerate impossibility condition $\mathcal{C}_f \cap \mathcal{V}_f \cap \mathcal{T}_0 = \emptyset$ established in Lemma 1, because $\mathcal{C}_f \cap \mathcal{V}_f$ may contain only rank-deficient elements (failing $\sigma_{\min}(T) \geq c_0$ or $\|T\|_{\mathrm{HS}} = 1$) that are excluded from $\mathcal{T}_0$. In such a regime, $\lambda_{\min}(A_f) = 0$ (so the spectral lower bound is vacuous), yet $\varepsilon^*_{\mathcal{T}_0} > 0$ may still hold. The two conditions characterize different aspects of the geometry: the spectral condition $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$ governs the global lower bound on A_f's quadratic form, while the relaxed-class condition governs whether any unit-norm non-degenerate operator can simultaneously satisfy both structural demands.

5.3 Geometric Interpretation: The Friedrichs Angle

For two closed subspaces $\mathcal{S}_1, \mathcal{S}_2$ of a Hilbert space $\mathcal{H}$, the *Friedrichs angle* $\theta_F(\mathcal{S}_1, \mathcal{S}_2) \in (0, \pi/2]$ is defined by

$$\cos\theta_F(\mathcal{S}_1, \mathcal{S}_2) := \sup\Big\{\langle u, v\rangle : u \in \mathcal{S}_1 \cap (\mathcal{S}_1 \cap \mathcal{S}_2)^\perp,\ v \in \mathcal{S}_2 \cap (\mathcal{S}_1 \cap \mathcal{S}_2)^\perp,\ \|u\| = \|v\| = 1\Big\},$$

with the convention $\theta_F = \pi/2$ if either of the slot subspaces is trivial. When $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$, the formula simplifies to $\cos\theta_F = \sup\{\langle u, v\rangle : u \in \mathcal{S}_1, v \in \mathcal{S}_2, \|u\| = \|v\| = 1\} = \|P_{\mathcal{S}_1} P_{\mathcal{S}_2}\|_{\mathrm{op}}$. Under this $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$ condition—the regime relevant to our application—the Friedrichs angle measures the geometric transversality of the two subspaces: θ_F near 0 indicates that $\mathcal{S}_1$ and $\mathcal{S}_2$ contain nearly aligned directions; $\theta_F = \pi/2$ indicates that the two subspaces are mutually orthogonal. (The general case in which $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \{0\}$ is more subtle: the slot subspaces remove the common direction, so a non-trivial intersection does not by itself imply $\theta_F = 0$. We do not use the general case in what follows.)

Proposition 5.1 (Geometric form of the lower bound)**.** *Under the hypotheses of Theorem 1, suppose in addition that $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$. Then*

$$\sqrt{\lambda_{\min}(A_f)} \geq \min\{\alpha\, \sigma^+_{\min}(L_f),\ \beta\} \cdot \sqrt{1 - \cos\theta_F(\mathcal{C}_f, \mathcal{V}_f)},$$

where $\sigma^+_{\min}(L_f)$ denotes the smallest non-zero singular value of L_f on $\mathcal{H}_{\mathrm{op}}$, and $\theta_F(\mathcal{C}_f, \mathcal{V}_f)$ is the Friedrichs angle between the coherent and verifiable subspaces in $\mathcal{H}_{\mathrm{op}}$.

The proof of Proposition 5.1 is given in Appendix A.4.

Accounting interpretation. The Friedrichs angle $\theta_F(\mathcal{C}_f, \mathcal{V}_f)$ measures the transversality of the institutional verification regime relative to the recirculation dynamics, in the operator-space metric induced by the Hilbert–Schmidt norm on $\mathcal{H}_{\mathrm{op}}$. When θ_F is small, the regime is *near-degenerate*: coherent and verifiable measurement maps nearly coincide, and the institutional tradeoff is weak. Such regimes correspond to environments in which the audit-evidence partitioning is closely aligned with the recirculation dynamics. When θ_F is large (approaching $\pi/2$), the regime is *strongly transversal*: the two subspaces are nearly orthogonal in operator space, and the institutional tradeoff is at its strongest.

The companion factor $\sigma_{\min}^{+}(L_f)$ measures the structural strength of the dynamics on $\mathcal{C}_f^{\perp}$. It is large when δ_Y and δ_R have well-separated spectral structures and small when their spectra are nearly degenerate. Section 7 will exhibit a regime parameter ρ that controls θ_F in a finite-dimensional toy model.

5.4 The Lower Bound and Its Limits

Three observations about the scope of the bound situate the constructive analysis of Section 6.

First, the bound is *structural* rather than constructive: it tells us that no admissible measurement map can drive the gap below $\sqrt{\lambda_{\min}(A_f)}$, but it does not exhibit a measurement map that attains the bound. Section 6 (Theorem 2) constructs a canonical measurement map W^* with explicit gap value $\alpha\|[\delta_Y, E_V]W^*_{\mathrm{coh}}P_R\|_{\mathrm{HS}}$. The relationship between the Theorem 2 gap value and the Theorem 1 lower bound is a separate question.

Second, the bound is *independent of any specific construction* of a candidate measurement map. The independence has a clear institutional reading: the coherence-verifiability conflict is a property of the institutional environment rather than of any specific firm-level measurement design. Institutional changes that reduce the bound—refinements of $\mathcal{G}$ that enlarge M_V (and hence the verifiable subspace $\mathcal{V}_f$), and structural alignments of δ_Y that reduce the spectral separation between the wealth-side and recirculation-side dynamics—are accordingly the only way to reduce the irreducible gap. In the extreme limit $\mathcal{G} = \mathcal{F}_Y$, $E_V = I_Y$ and $\mathcal{V}_f = \mathcal{H}_{\mathrm{op}}$, so $\mathcal{C}_f \subseteq \mathcal{V}_f$ and hence $\mathcal{C}_f \cap \mathcal{V}_f = \mathcal{C}_f \neq \{0\}$. By the positivity equivalence of Theorem 1, $\lambda_{\min}(A_f) = 0$ in this limit and the spectral lower bound vanishes. (Note that the Friedrichs-angle expression of Proposition 5.1 requires $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$ and is therefore not invoked in this limiting case; the positivity equivalence of Theorem 1 suffices to conclude $\lambda_{\min}(A_f) = 0$.) The implication for accounting policy is that measurement reform alone cannot achieve coherence and verifiability jointly; verification regime reform—specifically, refinement of the audit-evidence σ-algebra so that M_V encompasses more of the operator space accessible to admissible maps—is required.

Third, the bound treats the institutional weights (α, β) and the verification σ-algebra $\mathcal{G}$ as exogenous. The framework is silent on how these institutional parameters are determined.

6 Canonical Verifiability-Extreme Construction and the Residual Coherence Defect

This section turns to the constructive question: among admissible measurement maps with zero verifiability defect on the recirculation subspace, what is the structure of the residual coherence defect, and what does it reveal about the institutional source of Ijiri's impossibility?

The principal result, Theorem 2, exhibits an admissible measurement map W^* constructed as the composition of the conditional expectation onto the verification-supported subspace with a canonically chosen coherent measurement map. The result is best characterized not as an attainer of the Theorem 1 lower bound on the full weighted gap $\varepsilon = \alpha\varepsilon_{\mathrm{coh}} + \beta\varepsilon_{\mathrm{ver}}$—which it is not, in general, since the linear-weighted minimizer may sacrifice some verifiability for greater coherence—but as the canonical *verifiability-extreme* of the Pareto frontier between coherence and verifiability defects: the unique HS-orthogonal projection of a canonical coherent map onto the verifiable operator subspace, with explicit closed-form residual coherence defect.

The central interpretive contribution of the section concerns the closed-form expression of this residual:

$$\varepsilon_{\mathrm{coh}}(W^*) = \|[\delta_Y, E_V]\, W^*_{\mathrm{coh}} P_R\|_{\mathrm{HS}}.$$

The defect is the Hilbert–Schmidt norm of a single commutator: between the wealth-side differencing operator δ_Y and the verification projection E_V. Both objects entering the commutator are structural data of the institutional environment, independent of any choice of measurement map. The residual defect of the canonical verifiability-extreme map is thereby governed primarily by the structural relationship between δ_Y and E_V, with $W^*_{\mathrm{coh}} P_R$ entering as a normalized canonical orientation; the dominant source of the residual gap is institutional rather than merely firm-specific. This commutator characterization—rather than the lower-bound interpretation—is what we will identify as the principal contribution of the construction.

6.1 The Construction Strategy

We approach the constructive question through a two-step construction that prioritizes verifiability—selecting a coherent map first and then projecting it onto the verification-supported subspace. The choice to prioritize verifiability over coherence reflects a structural asymmetry between the two constraints. The verifiability constraint cuts out a closed linear subspace $\mathcal{V}_f = \mathcal{L}(R, M_V)$ of the operator space $\mathcal{H}_{\mathrm{op}}$, and conditional expectation E_V provides the canonical orthogonal projection onto this subspace. The coherence constraint, by contrast, is satisfied exactly only by the coherent subspace $\mathcal{C}_f \subseteq \mathcal{H}_{\mathrm{op}}$. Coherent maps form the kernel of the commutator operator L_f, and selection within this kernel requires resolving a structural ambiguity.

Step 1: A selected coherent representative. A coherent operator $T \in \mathcal{C}_f$ enjoys a structural freedom on the kernel of δ_R: for $f \in R$ with $\delta_R f = 0$, the coherence equation $\delta_Y T f = T \delta_R f = 0$ requires only that $Tf \in \ker \delta_Y$, and any choice within $\ker \delta_Y$ is admissible. To select a representative element of $\mathcal{C}_f$, we resolve this freedom by minimization:

$$W^*_{\mathrm{coh}}|_R \in \arg\min \{\|T\|_{\mathrm{HS}} : T \in \mathcal{C}_f,\ \sigma_{\min}(T) \geq c_0,\ \|T\|_{\mathrm{HS}} \geq 1\}.$$

However, since admissibility (Definition 3.3) is a property of measurement maps $W : H_Z \to H_Y$ defined on the full state space, not of restrictions $T : R \to H_Y$, we must require additionally that $W^*_{\mathrm{coh}}|_R$ admit a positive unital extension to all of H_Z. The following assumption captures this requirement.

Assumption 6.1 (Existence of a selected coherent representative)**.** There exists an admissible measurement map $W^*_{\mathrm{coh}} \in \mathcal{W}$ such that:

- the restriction $T^*_{\mathrm{coh}} := W^*_{\mathrm{coh}}|_R$ lies in $\mathcal{C}_f$ (coherence on R, with range in M_f);
- $\|W^*_{\mathrm{coh}} P_R\|_{\mathrm{HS}} = 1$ and $\sigma_{\min}(W^*_{\mathrm{coh}}|_R) \geq c_0$ (non-degeneracy);

- T^*_{coh} is an HS-norm-minimizing element of the operator-space class $\mathcal{C}_f \cap \{T \in \mathcal{H}_{\mathrm{op}} : \sigma_{\min}(T) \geq c_0,\ \|T\|_{\mathrm{HS}} \geq 1\}$.

Remark 6.1 (On the use of "canonical"). We use the adjective "canonical" for W^*_{coh} as a label for "selected via the HS-norm minimization criterion of Assumption 6.1," rather than as a claim of unique determination. The criterion fixes $\|T^*_{\mathrm{coh}}\|_{\mathrm{HS}} = 1$ but may admit multiple unit-norm coherent representatives (e.g., when δ_R has repeated eigenvalues or δ_Y admits additional symmetries). Theorem 2 part (5) makes this explicit: W^* is determined relative to a chosen representative, not in absolute terms. A stronger uniqueness criterion (e.g., minimization of distance from a reference representation) could be adopted but is not required for the closed-form residual defect of Theorem 2 part (3).

In the finite-dimensional setting of Assumption 5.1, the operator-space minimization reduces to a Sylvester-type matrix equation. We impose the existence of a positive unital extension as a substantive feasibility assumption: characterizing exactly when an operator-space minimizer $T^*_{\mathrm{coh}} \in \mathcal{H}_{\mathrm{op}}$ admits a positive unital extension $W^*_{\mathrm{coh}} : H_Z \to H_Y$ depends on the order structure of $R \subset H_Z$ in addition to spectral data, and a complete characterization is beyond the scope of the present paper. The institutional content of the assumption is that the wealth-side dynamics, restricted to the range of a candidate measurement, must be capable of supporting the period-to-period evolution of recirculation streams while admitting a positivity- and constant-preserving extension. A degenerate institutional setting in which this condition fails would correspond to wealth-side dynamics too constrained to accommodate the recirculation structure of R admissibly—an extreme case we exclude.

Step 2: The verifiability projection. Define the *canonical verifiability-extreme map* as $W^* := E_V \circ W^*_{\mathrm{coh}}$. This composition has range in M_V by construction. The coherence constraint is generally violated.

6.2 Theorem 2

Theorem 2 (Canonical verifiability-extreme map)**.** *Under Assumptions 3.1, 3.2, 5.1, and 6.1, the map $W^* := E_V \circ W^*_{\mathrm{coh}}$ satisfies:*

(1) ***Admissibility.*** *$W^* \in \mathcal{W}$. That is, W^* is linear, bounded, positivity-preserving, and constant-preserving.*

(2) ***Zero verifiability defect.*** $\varepsilon_{\mathrm{ver}}(W^*) = 0$.

(3) ***Closed-form coherence defect.*** *The coherence defect of W^* admits the explicit expression*

$$\varepsilon_{\mathrm{coh}}(W^*) = \|[\delta_Y, E_V]\, W^*_{\mathrm{coh}}\, P_R\|_{\mathrm{HS}}.$$

(4) ***Canonical-ness as projection.*** *Let $T^*_{\mathrm{coh}} := W^*_{\mathrm{coh}}|_R \in \mathcal{H}_{\mathrm{op}}$ be the restriction of the canonical coherent map to the recirculation subspace (taking values in M_f by Assumption 5.1). The element $T^* := E_V \circ T^*_{\mathrm{coh}} \in \mathcal{V}_f$ is the unique HS-orthogonal projection of T^*_{coh} onto the*

verifiable operator subspace $\mathcal{V}_f$:

$$T^* = \arg\min_{T \in \mathcal{V}_f} \|T - T^*_{\mathrm{coh}}\|_{\mathrm{HS}}.$$

(5) ***Determinacy modulo coherent selection.*** *Once a canonical coherent representative W^*_{coh} has been selected (Assumption 6.1), the construction $W^* = E_V \circ W^*_{\mathrm{coh}}$ is uniquely determined. Any non-uniqueness in W^* arises from non-uniqueness in the canonical coherent selection itself; one identifiable source of such non-uniqueness is the kernel ambiguity on $R \cap \ker \delta_R$, where the coherence equation places no constraint on the value of W^*_{coh}.*

The proof is given in Appendix A.5. The five parts merit separate brief comment.

Part (1) establishes that W^* qualifies as an admissible measurement map. The verification of admissibility relies on three properties of the conditional expectation E_V: bounded with operator norm 1; preserves positivity; and preserves constants. Composition with the admissible W^*_{coh} inherits each property. The verification depends critically on the modeling choice of Section 3.3, in which the verification projection was identified as a conditional expectation rather than an arbitrary orthogonal projection. An arbitrary closed subspace $M_V \subseteq H_Y$ admits an orthogonal projection P_{M_V}, but P_{M_V} in general does not preserve positivity. The conditional-expectation specification is therefore not a technical convenience but a structural prerequisite for an admissibility-preserving construction.

Part (2) is direct: the range of W^* on H_Z is $E_V(W^*_{\mathrm{coh}}(H_Z)) \subseteq M_V$.

Part (3) is the central technical content. Its proof rests on a substitution chain that exploits both the coherence of W^*_{coh} and the verifiability structure of E_V, producing the commutator $[\delta_Y, E_V]$.

Part (4) is the canonical-ness claim. We emphasize that part (4) does not assert that W^* minimizes ε over the entire admissible class $\mathcal{W}_0$. The construction W^* delivers the $\mathcal{V}_f$-extreme of the Pareto frontier between $\mathcal{C}_f$ and $\mathcal{V}_f$.

Part (5) records the residual freedom: once a canonical coherent map is selected, W^* is determined; non-uniqueness in W^* traces back to non-uniqueness in the coherent selection. The kernel ambiguity on $R \cap \ker \delta_R$ is one identifiable source of such non-uniqueness, but other sources may arise depending on the spectral structure of δ_Y and δ_R.

6.3 The Commutator Characterization

Part (3) of Theorem 2 yields the central interpretive contribution of the section.

The coherence defect of the canonical verifiability-extreme map W^* is the Hilbert–Schmidt norm of a single object: the operator $[\delta_Y, E_V] W^*_{\mathrm{coh}} P_R$. Two of the three factors in this product depend only on structural data: δ_Y is the wealth-side differencing operator; E_V is the conditional expectation onto the verification σ-algebra; their commutator $[\delta_Y, E_V]$ is therefore determined entirely by the relationship between these two structures, independently of any choice of measurement map. The third factor, $W^*_{\mathrm{coh}} P_R$, is the selected coherent representative restricted to the recirculation subspace.

The force of this characterization is accounting-specific. The projection E_V is not an arbitrary orthogonal projection onto a closed subspace of H_Y. It is the conditional expectation induced by an audit-evidence σ-algebra and therefore preserves positivity and constants, as required for admissible accounting measurement. Likewise, the defect is not evaluated on an arbitrary domain but on the recirculation subspace R, where accrual reversals, deferred items, and recyclable components cycle between stock-domain and flow-domain representation. Thus the commutator $[\delta_Y, E_V]$ is interpreted not merely as a formal operator expression but as the failure of audit-evidence partitions to be preserved by wealth-side accounting dynamics.

When δ_Y commutes with E_V—when the wealth-side differencing operator preserves verification σ-algebra measurability, so that the differencing of a verifiable wealth reading is itself verifiable, and the differencing of an unverifiable wealth reading remains unverifiable—the commutator vanishes, the coherence defect of W^* vanishes, and W^* achieves both coherence and verifiability simultaneously. When the commutator is non-zero *on the canonical coherent range*, i.e., when $[\delta_Y, E_V]W^*_{\text{coh}}P_R \neq 0$, the residual coherence defect is positive, and Ijiri's impossibility result is forced by a specific structural tension: between the temporal evolution of wealth and the partitioning of audit-relevant information. (We note for precision that $[\delta_Y, E_V] \neq 0$ as a global operator does not by itself imply a positive defect; the defect formula evaluates the commutator on the range of $W^*_{\text{coh}}P_R$, and could in principle vanish if that range happened to lie in $\ker[\delta_Y, E_V]$.)

This characterization sharpens Ijiri (1986)'s original framing. Ijiri located the source in $[\delta, W] \neq 0$—the non-commutativity of the differencing operator with the measurement map—a property that depends on the choice of W. Theorem 2 part (3) shows that the dominant structural source of the residual gap involves objects that are both structural data of the institutional environment: δ_Y and E_V. The remaining factor $W^*_{\text{coh}}P_R$ enters as a normalized canonical orientation, fixed at unit Hilbert–Schmidt norm by Definition 3.4, but its choice may still influence the magnitude of the residual defect along directions exposed to the commutator. *The dominant source of the impossibility is institutional rather than merely firm-specific:* the commutator $[\delta_Y, E_V]$ identifies the institutional component of the residual gap, while the firm-level measurement design enters only through the normalized orientation of the canonical coherent map.

Accounting interpretation. The commutator $[\delta_Y, E_V]$ measures the failure of the verification structure to be preserved under dynamics: a state whose value is verifiable today need not have a verifiable value next period. When the institutional verification regime is rich enough—when audit evidence partitions are stable across accounting periods—the commutator vanishes and the coherence-verifiability conflict dissolves. When the verification regime is structured around evidence partitions that drift relative to wealth dynamics, the commutator has non-trivial action on the canonical coherent range $W^*_{\text{coh}}(R)$. The width of the gap is in this sense a quantitative measure of audit-information conservatism—the degree to which the institutional partitioning of evidence fails to track the underlying movement of accounting states.

6.4 Comparative Statics, Scope, and What Theorem 2 Does Not Imply

Comparative statics on the verification regime. Refinements of $\mathcal{G}$ enlarge M_V and bring E_V closer to the identity. In the limit $\mathcal{G} = \mathcal{F}_Y$, $E_V = I_Y$ and $[\delta_Y, E_V] = 0$, the coherence defect of W^* vanishes. Coarsenings of $\mathcal{G}$ shrink M_V and generally enlarge the commutator, hence the coherence defect.

Comparative statics on wealth-side dynamics. Operators δ_Y that approximately preserve $\mathcal{G}$ yield small coherence defect. Operators that mix substantially across the partition $\mathcal{G}$ yield larger defect.

Comparative statics on the recirculation subspace. Larger and more complex recirculation subspaces need not change $\|W^*_{\mathrm{coh}} P_R\|_{\mathrm{HS}}$, which is fixed at unit value by the normalization of Definition 3.4. Rather, they can increase the residual coherence defect by exposing more directions of R to the non-commuting component of $[\delta_Y, E_V]$: a higher-dimensional or richer recirculation structure provides more channels through which the failure of δ_Y to commute with E_V can manifest as a coherence defect.

We close the section with four observations about what Theorem 2 does not imply.

First, W^* is not the global minimizer of ε over $\mathcal{W}_0$. The construction selects the $\mathcal{V}_f$-extreme of the Pareto frontier. The minimizer of the linear-weighted gap $\alpha\varepsilon_{\mathrm{coh}} + \beta\varepsilon_{\mathrm{ver}}$ may lie at non-extreme points of the frontier.

Second, Theorem 2 establishes admissibility of W^* in the class $\mathcal{W}$ (Definition 3.3: linearity, boundedness, positivity preservation, constant preservation), but does not assert membership in the non-degenerate class $\mathcal{W}_0$. Specifically, the projection by E_V in the construction $W^* = E_V \circ W^*_{\mathrm{coh}}$ can in principle reduce rank or Hilbert–Schmidt scale on R, so the unit-norm condition $\|W^* P_R\|_{\mathrm{HS}} = 1$ and the singular-value condition $\sigma_{\min}(W^*|_R) \geq c_0$ need not be inherited from W^*_{coh}. Membership in $\mathcal{W}_0$ would require additional non-degeneracy conditions on $E_V T^*_{\mathrm{coh}}$ that we do not impose. The central question (Q) is thus posed over $\mathcal{W}_0$, but Theorem 2 addresses a structurally meaningful neighbouring construction in $\mathcal{W}$, with the choice of class motivated by the goal of obtaining an explicit closed-form residual defect.

Third, Theorem 2 does not pursue the symmetric construction with priorities reversed. The symmetric construction is best treated separately.

Fourth, Theorem 2 treats the canonical coherent map W^*_{coh} as the operative object; alternative canonical selections within $\mathcal{C}_f$ may yield different gap values. The investigation of alternative conventions is left to future work.

7 Analytic Reconstruction of the 1986–1990 Momentum Definition Shift

This section applies the framework developed in Sections 3–6 to the historical question that opened the paper: why did Ijiri move, between 1986 and 1990, from a wealth-based to an income-based definition of accounting momentum? The application proceeds through analytic reconstruction in a finite-dimensional toy model, in which the recirculation subspace, the wealth-side dynamics, and the verification regime can all be specified explicitly, and the gap functional can be evaluated in closed form for both the wealth-based and the income-based measurement maps.

The principal result of the section, Proposition 7.1, is a *regime-conditional* comparative inequality: the relative ordering of $\varepsilon(W_{1986})$ and $\varepsilon(W_{1990})$ depends on a single regime parameter ρ that encodes the alignment of the verification-supported subspace with the income-side versus the stock-side dimensions of wealth. Above a structural threshold ρ^*, the income-based map W_{1990} has the smaller gap; below it, the wealth-based map W_{1986} does. Ijiri (1990)'s revision is thereby placed within a theoretical structure that explains its suitability for one institutional regime and its potential limitations in regimes structured around fair-value or stock-position evidence.

7.1 The 1986–1990 Definitional Shift

The definition of accounting momentum that Ijiri (1986) introduced and that he revised over the following four years rests on a substantive ambiguity about which side of the accounting cycle—stock or flow—provides the underlying state variable for the rate-of-change concept. In Ijiri (1986), momentum was identified with the time derivative of wealth: a stock-domain quantity, anchored in balance-sheet positions, whose temporal evolution generated the flows that appeared in the income statement. The framework treated accounting flows as derivatives of stock-side state variables, and momentum as a second-order quantity capturing the rate of change of those derivatives. This framing placed momentum within a stock-first ontology of accounting, in which the income statement was a derived artifact of balance-sheet evolution.

By 1990, in Ijiri (1990), Ijiri had reformulated momentum as the time derivative of income. Income, no longer treated as a derivative of wealth, became the underlying state variable, with momentum capturing its rate of change. Wealth, in turn, was conceptualized as the integral of income over time. The framework shifted to a flow-first ontology of accounting, in which the balance sheet was a derived artifact of accumulated income.

Ijiri offered no extended discussion of why he moved between the two definitions. Accounting historians have variously treated the change as a presentational evolution, an alignment with prevailing reporting conventions, or a refinement under the influence of contemporary accruals-based accounting research; none of these accounts treats the shift as a substantive theoretical commitment. Yet the change is not cosmetic. A wealth-based momentum and an income-based momentum imply different relationships between balance-sheet and performance measurement, different roles for the verification structure of audit evidence, and different ontologies of recirculation through other comprehensive income.

The hypothesis we develop in this section is that Ijiri's shift reflects a structural commitment to a particular institutional verification regime—specifically, the income-centric audit regime that prevailed in the North-American institutional context of the late 1980s. Under that regime, we will show, the income-based momentum W_{1990} yields the smaller coherence-verifiability gap; under a stock-centric audit regime, the wealth-based momentum W_{1986} does. The 1990 revision is then best understood not as a universal refinement of the 1986 framework but as an institutional alignment.

7.2 An Analytic Setting

We specialize the framework of Section 3 to a finite-dimensional setting that admits closed-form computation while preserving the structural features relevant to the comparison.

The recirculation subspace. Take the recirculation subspace as the direct sum of three one-dimensional components:

$$R = R_A \oplus R_D \oplus R_O,$$

where $R_A = \mathrm{span}\{r_A\}$ corresponds to accrual reversals, $R_D = \mathrm{span}\{r_D\}$ to deferred items, and $R_O = \mathrm{span}\{r_O\}$ to OCI-type recyclables. The dimension of R is $r = 3$. The state-side differencing operator on R is diagonal in the canonical basis: $\delta_R = \mathrm{diag}(\lambda_A, \lambda_D, \lambda_O)$, where $\lambda_A, \lambda_D, \lambda_O \in \mathbb{R}$ encode the period-to-period dynamics of each recirculation channel.

The wealth/income reading space. Mirroring the three components of R, the wealth space H_Y has a paired structure: for each component $k \in \{A, D, O\}$, H_Y contains a two-dimensional "wealth slot" $Y_k = \mathrm{span}\{y^k_{\mathrm{inc}}, y^k_{\mathrm{stock}}\}$, with y^k_{inc} representing the income-aligned wealth reading for component k and y^k_{stock} representing the stock-aligned reading. The full wealth space is $H_Y = Y_A \oplus Y_D \oplus Y_O$, of dimension 6. The wealth-side differencing operator is block-diagonal with respect to this decomposition: δ_Y maps each Y_k to itself, with action

$$\delta_Y|_{Y_k} = \begin{pmatrix} \mu_{\mathrm{inc}} & 0 \\ 0 & \mu_{\mathrm{stock}} \end{pmatrix}$$

in the $\{y^k_{\mathrm{inc}}, y^k_{\mathrm{stock}}\}$ basis. We assume that the wealth-side dynamics are homogeneous across components—$\mu_{\mathrm{inc}}, \mu_{\mathrm{stock}}$ are independent of k—to isolate the regime parameter from the spectral details of individual components. The general inhomogeneous case is treated in Appendix B.

The verification structure (evidence-projection analogue). The institutional verification regime is specified through a single parameter $\rho \in [0, 1]$ that controls the alignment of the verification-supported subspace M_V with the income-aligned versus the stock-aligned dimensions of wealth. For each component k, the k-th wealth slot Y_k contributes a one-dimensional subspace

$$M^k_V(\rho) := \mathrm{span}\{\rho \cdot y^k_{\mathrm{inc}} + \sqrt{1-\rho^2} \cdot y^k_{\mathrm{stock}}\}$$

to the verification-supported subspace. The full subspace is $M_V(\rho) = M^A_V(\rho) \oplus M^D_V(\rho) \oplus M^O_V(\rho)$, of dimension 3. We denote by $P_V(\rho) : H_Y \to M_V(\rho)$ the orthogonal projection. We use the notation $P_V(\rho)$ rather than $E_V(\rho)$ throughout Section 7 to emphasize that this is a stylized orthogonal projection (in general not a conditional expectation), distinct from the conditional expectation $E_V = \mathbb{E}[\,\cdot\,|\mathcal{G}]$ used in Sections 3–6.

Remark 7.1 (Evidence-projection analogue rather than literal conditional expectation)*.* We emphasize that $M_V(\rho)$ as constructed above is a *stylized evidence projection* rather than the range of a conditional expectation onto a sub-σ-algebra in the strict sense of Section 3.3. The range of a conditional expectation $E[\,\cdot\,|\mathcal{G}]$ on $L^2(\mathcal{Y}, \mathcal{F}_Y, \nu)$ is the space of $\mathcal{G}$-measurable functions, which always contains the constant functions; for generic $\rho \in (0,1)$, the rotated one-dimensional subspace $M_V^k(\rho)$ does not arise as such a range. The toy model thus uses an *evidence-projection analogue*—an orthogonal projection onto a finite-dimensional subspace of H_Y that captures the qualitative structure of an income-versus-stock verification regime—rather than a literal conditional expectation. This analogue is sufficient to illustrate the operator-space geometry of the framework and the regime-conditional ordering of W_{1986} and W_{1990}. A toy model in which $P_V(\rho)$ is literally a conditional expectation for every ρ would require a continuous family of sub-σ-algebras producing rotated range subspaces, which is not generic. The conceptual content of Theorem 2—specifically, the commutator characterization $\varepsilon_{\text{coh}}(W^*) = \|[\delta_Y, E_V]W^*_{\text{coh}}P_R\|_{\text{HS}}$—continues to hold formally with E_V replaced by any orthogonal projection of H_Y onto a closed subspace (such as $P_V(\rho)$), but the admissibility of $W^* = E_V \circ W^*_{\text{coh}}$ in the strict sense of Definition 3.3 is not automatic outside the conditional-expectation case. The reader should accordingly read Section 7 as a structural illustration of the framework's geometry rather than as a strict instance of Theorem 2's admissibility-preserving construction.

The parameter ρ admits a direct institutional interpretation. When $\rho = 1$, M_V consists entirely of income-aligned wealth readings—the verification regime supports independently determinable evidence about income-side measurements but not about stock-side measurements. This corresponds to an *income-dominant verification regime*. When $\rho = 0$, M_V consists entirely of stock-aligned readings—a *stock-dominant verification regime*. Intermediate values $\rho \in (0,1)$ correspond to mixed regimes, with $\rho = 1/\sqrt{2}$ representing the balanced point.

7.3 The Two Measurement Maps

The wealth-based map W_{1986}. The 1986 definition treats momentum as the time derivative of wealth, with wealth understood as a stock-side quantity. We define

$$W_{1986}(r_k) := \tfrac{1}{\sqrt{3}} y^k_{\text{stock}} \quad \text{for } k \in \{A, D, O\},$$

or in matrix form, $W_{1986} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix} \in \mathbb{R}^{6\times 3}$.

The income-based map W_{1990}. The 1990 definition treats momentum as the time derivative of income:

$$W_{1990}(r_k) := \tfrac{1}{\sqrt{3}} y^k_{\text{inc}} \quad \text{for } k \in \{A, D, O\},$$

or in matrix form, $W_{1990} = \frac{1}{\sqrt{3}} \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix} \in \mathbb{R}^{6\times 3}$.

The factor $1/\sqrt{3}$ enforces unit Hilbert–Schmidt norm: each map's three columns are mutually orthonormal in $\mathbb{R}^6$ before scaling, giving $\|W\|^2_{\text{HS}} = 3$ pre-normalization, and division by $\sqrt{3}$ yields $\|W_{1986}\|_{\text{HS}} = \|W_{1990}\|_{\text{HS}} = 1$. Both maps thereby satisfy the unit-norm condition of Definition 3.4, with smallest singular value $1/\sqrt{3}$, so they are non-degenerate candidate maps for any $c_0 \leq 1/\sqrt{3}$ (which respects the feasibility constraint (1) for $r = 3$). Their full admissibility (positivity preservation, constant preservation) on the entire state space H_Z is not relevant to the gap comparison, which depends only on the operator-space structure of $W|_R$.

For the institutional reading developed in Section 7.5, W_{1986} and W_{1990} are best read as natural representatives of the stock-first and flow-first ontologies of accounting measurement that Ijiri considered, with their full state-space behavior taken as fixed by the broader institutional context.

Closed-form gap values. Direct computation yields explicit expressions. The coherence defect of W_{1986} arises from the mismatch between the stock-side wealth dynamics and the recirculation dynamics:

$$\varepsilon_{\text{coh}}(W_{1986}) = \tfrac{1}{\sqrt{3}}\sigma_{\text{stock}}, \quad \text{where } \sigma_{\text{stock}} := \sqrt{(\mu_{\text{stock}} - \lambda_A)^2 + (\mu_{\text{stock}} - \lambda_D)^2 + (\mu_{\text{stock}} - \lambda_O)^2}.$$

By parallel computation,

$$\varepsilon_{\text{coh}}(W_{1990}) = \tfrac{1}{\sqrt{3}}\sigma_{\text{inc}}, \quad \text{where } \sigma_{\text{inc}} := \sqrt{(\mu_{\text{inc}} - \lambda_A)^2 + (\mu_{\text{inc}} - \lambda_D)^2 + (\mu_{\text{inc}} - \lambda_O)^2}.$$

The verifiability defects involve ρ. For W_{1986}, the image lies entirely in the stock-aligned subspace; the projection of each (normalized) basis vector $\tfrac{1}{\sqrt{3}}y^k_{\text{stock}}$ onto $M^k_V(\rho)$ recovers a magnitude $\tfrac{1}{\sqrt{3}}\sqrt{1-\rho^2}$ with residual magnitude $\tfrac{1}{\sqrt{3}}\rho$; the contribution to the verifiability defect from each component is $\rho/\sqrt{3}$, and the total defect is $\varepsilon_{\text{ver}}(W_{1986}) = \rho$. By analogous calculation, $\varepsilon_{\text{ver}}(W_{1990}) = \sqrt{1-\rho^2}$.

The total gap functionals are accordingly

$$\begin{aligned} \varepsilon(W_{1986}) &= \tfrac{\alpha}{\sqrt{3}}\,\sigma_{\text{stock}} + \beta\rho, \\ \varepsilon(W_{1990}) &= \tfrac{\alpha}{\sqrt{3}}\,\sigma_{\text{inc}} + \beta\sqrt{1-\rho^2}. \end{aligned} \tag{3}$$

7.4 The Regime-Conditional Ordering

The comparison between $\varepsilon(W_{1986})$ and $\varepsilon(W_{1990})$ reduces to an algebraic inequality in ρ, parametrized by the *coherence asymmetry*

$$\Delta := \sigma_{\text{inc}} - \sigma_{\text{stock}}$$

and by the institutional weights (α, β).

Proposition 7.1 (Regime-conditional comparative ordering). *In the analytic setting of Sections 7.2–7.3, define*

$$k := \frac{\alpha\Delta}{\beta\sqrt{3}}, \qquad \rho^*(k) := \frac{k + \sqrt{2-k^2}}{2}, \quad k \in [-1, 1].$$

Then:

(a) For $k \in [-1,1]$, the threshold $\rho^*(k) \in [0,1]$ is the unique value of ρ at which $\varepsilon(W_{1986}) = \varepsilon(W_{1990})$.

(b) For $\rho > \rho^*(k)$, the income-based map W_{1990} has the smaller gap: $\varepsilon(W_{1990}) < \varepsilon(W_{1986})$.

(c) For $\rho < \rho^*(k)$, the wealth-based map W_{1986} has the smaller gap: $\varepsilon(W_{1986}) < \varepsilon(W_{1990})$.

(d) In the spectrally symmetric case $\Delta = 0$, the threshold reduces to $\rho^*(0) = 1/\sqrt{2}$.

(e) For $|k| > 1$, no value of $\rho \in [0, 1]$ equalizes the two gaps; one map dominates the other across all admissible verification regimes. Specifically, $k > 1$ implies $\varepsilon(W_{1986}) < \varepsilon(W_{1990})$ for all $\rho \in [0, 1]$ (the wealth-based map dominates), while $k < -1$ implies $\varepsilon(W_{1990}) < \varepsilon(W_{1986})$ for all $\rho \in [0, 1]$ (the income-based map dominates).

The proof of Proposition 7.1 is given in Appendix A.6.

The threshold $\rho^*(k)$ is monotonically increasing in k: spectral asymmetries that disfavor income-side dynamics push the threshold above $1/\sqrt{2}$; spectral asymmetries that favor income-side dynamics push it below. The spectrally symmetric case $\Delta = 0$ yields the balanced threshold $\rho^* = 1/\sqrt{2}$: the regime-parameter midpoint between income-dominance and stock-dominance is exactly the indifference point.

Connection to Section 5. The regime parameter ρ controls the Friedrichs angle $\theta_F(\mathcal{C}_f, \mathcal{V}_f)$ of Section 5.3 in a direct way. As ρ varies in $[0, 1]$, the verification-supported subspace $M_V(\rho)$ rotates within H_Y, and the operator-space subspace $\mathcal{V}_f = \mathcal{L}(R, M_V) \subseteq \mathcal{H}_{\mathrm{op}}$ rotates correspondingly within the working space. The coherent subspace $\mathcal{C}_f$ is fixed by the spectral data and does not depend on ρ. The Friedrichs angle $\theta_F(\rho) := \theta_F(\mathcal{C}_f, \mathcal{V}_f(\rho))$ is therefore a function of ρ, as is the Theorem 1 lower bound $\sqrt{\lambda_{\min}(A_f(\rho))}$. The threshold ρ^* does not in general coincide with the value of ρ that minimizes the Theorem 1 lower bound; rather, it locates the regime in which the two specific historical maps trade dominance.

7.5 Implications for Ijiri's Definitional Shift

Primary reading. The North-American audit and reporting environment in which Ijiri was working in the late 1980s was structured around accruals-based income measurement, with audit evidence organized predominantly around realized income, accrual reversal, and periodized performance verification. The conservatism principle described by Watts (2003a,b), the asymmetric-timeliness characterization of Basu (1997), and the historical role of accruals-based earnings as the locus of audit attestation (Bushman and Smith, 2001) are representative features of this regime. In the analytic framework of the present paper, that environment is naturally represented, within the toy model, by a regime with ρ plausibly above $1/\sqrt{2}$—an income-dominant regime. Conditional on that institutional reading and within the scope of the toy model, Proposition 7.1 implies $\varepsilon(W_{1990}) < \varepsilon(W_{1986})$.

The hypothesis is that Ijiri's revision between 1986 and 1990 reflects this conditional advantage: the move from a wealth-based to an income-based momentum is best understood not as a universal refinement of the 1986 framework but as an institutional alignment, a measurement choice that achieves a smaller residual gap conditional on the income-dominant verification regime under which Ijiri's framework was to operate. The 1986 framework, in this reading, was internally coherent and correct as a stock-first ontology of accounting measurement, but has the larger residual gap under the institutional context in which Ijiri sought to deploy it.

Secondary reading: contemporary institutional contexts. The institutional environment of accounting in the early twenty-first century has shifted: the rise of fair-value measurement under IFRS 9 and IFRS 13 (Whittington, 2008; Magnan et al., 2015; Lawrence et al., 2016), the expansion of OCI recycling and its surrounding debate (Black, 2016; Bradbury, 2016; Linsmeier, 2016), and the development of audit standards centered on stock-position attestation rather than periodic income certification (Song et al., 2010)—all push the contemporary verification regime toward smaller ρ values. Under such regimes, Proposition 7.1 implies the reverse inequality: $\varepsilon(W_{1986}) < \varepsilon(W_{1990})$. The wealth-based momentum that Ijiri used in 1986 may, in fact, occupy the more efficient position among the two historical candidate maps under the contemporary institutional verification regime than the income-based momentum he adopted in 1990.

This is not a recommendation to revert to the 1986 framework. The choice of measurement map in any specific institutional setting depends on factors beyond the gap functional. What the analysis does suggest is that Ijiri's 1990 revision should not be read as a universal theoretical advance over the 1986 framework. The two are, structurally, regime-conditional alternatives.

Limitations of the analytic reconstruction. Three limitations warrant explicit acknowledgment.

First, the toy model is stylized. Real accounting environments involve recirculation structures of higher dimension than three, with components of varying sign and magnitude in their dynamics; verification σ-algebras whose alignment with income and stock dimensions is more nuanced than a single regime parameter; and institutional weights that are themselves contested and time-varying. The closed-form expressions of (3) are exact in the toy model but illustrative, rather than predictive, of real institutional comparisons.

Second, the comparison conditions on the institutional verification regime ρ as exogenous data, not on the recirculation structure R or on the spectral data as choice variables.

Third, the analytic reconstruction is not historical evidence about Ijiri's actual reasoning. We do not claim that Ijiri explicitly considered the gap functional, the regime parameter ρ, or the threshold ρ^*. What we offer is a *structural rationalization*: a theoretical account of why the 1990 revision was institutionally well-suited to the environment in which Ijiri was working. The reconstruction is offered in the spirit of accounting thought history that uses contemporary formal frameworks to illuminate the structural content of past theoretical commitments.

8 Conclusion

8.1 Summary

This paper has developed a constructive variational reconstruction of Ijiri's measurement problem. Beginning from Ijiri's (1986) impossibility insight—that on the recirculation subspace of an accounting state space, no measurement architecture can simultaneously satisfy coherence and verifiability—we have transformed the impossibility into a quantitative theory of measurement design under verification constraints. Theorem 1 establishes a structural lower bound on the coherence–verifiability gap, expressed in terms of the Friedrichs angle between the coherent and verifiable subspaces in operator space. Theorem 2 constructs a canonical verifiability-extreme map W^* and characterizes its residual coherence defect as the Hilbert–Schmidt norm of a single commutator: $[\delta_Y, E_V]$ applied to the canonical coherent map. The commutator characterization yields what we have proposed as the central interpretive insight of the analysis: *the dominant structural source of the residual coherence–verifiability gap is institutional rather than merely firm-specific*—specifically, the relationship between wealth-side dynamics δ_Y and the verification σ-algebra $\mathcal{G}$. The choice of W^*_{coh} enters as a normalized orientation on the recirculation subspace; the institutional commutator $[\delta_Y, E_V]$ determines which orientations are exposed to residual defect, and both of its factors are structural data of the institutional environment. Application of the framework to Ijiri's 1986–1990 momentum definition shift yields a regime-conditional comparative inequality: above a threshold value of the regime parameter ρ, the income-based momentum has the smaller gap; below it, the wealth-based momentum does. Ijiri's revision is thereby placed within a theoretical structure that explains its institutional well-suitedness without claiming universal theoretical advance.

8.2 Limitations

Five limitations of the framework warrant explicit acknowledgment.

First, the framework is restricted throughout to finite-dimensional recirculation subspaces and finite-rank measurement maps. The infinite-dimensional case—in which R is countably or uncountably infinite-dimensional and W is a general bounded operator—is sketched in Appendix B but not developed. The principal technical obstacle is the loss of automatic compactness of the unit ball in $\mathcal{T}_0$, which would require a more delicate analysis to establish attainment of the lower bound.

Second, the verification structure $\mathcal{G}$ is treated as exogenous data. A complete theory of accounting measurement design would endogenize $\mathcal{G}$ as the choice variable of a regulator or standards-setter optimizing over admissible verification regimes. Such an endogenization is the natural extension of the present framework but is beyond the scope of this paper.

Third, the institutional weights (α, β) are similarly treated as exogenous. A more complete framework would model these weights as the outputs of an institutional bargaining process between the demands of audit conservatism and the demands of timely information.

Fourth, the dynamics of the verification structure are static. A time-indexed verification filtration $\{\mathcal{G}_t\}_{t\in T}$ would yield a dynamic version of the framework in which E_V varies over time and the commutator $[\delta_Y, E_V]$ becomes time-dependent. The dynamic extension is mathematically natural but conceptually substantial.

Fifth, the analytic reconstruction of Ijiri's 1986–1990 shift is a structural rationalization, not historical evidence about Ijiri's reasoning. The framework predicts that the 1990 revision was institutionally well-suited to the late-1980s North-American audit environment; it does not establish that Ijiri's actual reasoning followed the lines of the framework.

8.3 Open Directions

Four open directions are particularly worth noting.

The first is standards-setting as uncertainty management. Treating $\mathcal{G}$ as the choice variable of a regulator, with the regulator's objective involving both the firm-level gap $\varepsilon(W^*)$ and a separate "information cost" of maintaining a fine σ-algebra, yields a standards-setting variant of the framework in which the optimal verification regime is itself an object of analysis. In this formulation, a standard setter does not simply choose a preferred measurement attribute; it manages an irreducible uncertainty generated by the non-commuting relationship between accounting dynamics and audit-evidence partitioning. The direction connects naturally to the standards-setting paper that completes the three-paper sequence to which the present paper belongs.

The second is the dynamic extension. A time-indexed verification filtration $\{\mathcal{G}_t\}$ would allow analysis of how the coherence–verifiability gap evolves over the accounting period, how it responds to refinements of audit information, and how the canonical verifiability-extreme map W_t^* varies as the verification regime evolves. The framework would connect to the literature on dynamic information acquisition and to work on path-dependent measurement (Suzuki, 2026a).

The third is the empirical operationalization of the framework. The regime parameter ρ, the spectral asymmetry Δ, and the institutional weights (α, β) are all in principle estimable from institutional data on audit standards, recirculation structures, and reporting requirements. An empirical implementation would allow the analytic reconstruction of Section 7 to be tested against the actual institutional history of momentum-based measurement frameworks, including their reception and modification across different national audit environments.

The fourth is a broader uncertainty program for accounting measurement. The analogy is structural rather than physical: just as uncertainty relations in quantum theory gave rise to questions about optimal states, measurement disturbance, joint measurability, and informational versions of uncertainty, Theorems 1 and 2 suggest accounting analogues. One direction is an accounting measurement-disturbance theory, asking how much coherence is disturbed when verification is imposed through a projection such as E_V. A second is an entropic coherence–verifiability uncertainty theory, replacing Hilbert–Schmidt defects with informational measures of unverifiable or dynamically incoherent residuals. These extensions would not alter the accounting-specific content of the present paper; they would extend the same core problem—measurement under non-commuting institutional constraints—to new objective functions and institutional settings.

The paper has not solved the coherence–verifiability problem, nor offered a recommendation for accounting standards. What it has offered is a quantitative theory of when the irreducible measurement gap is large, when it is small, and which measurement designs sit at which positions on the Pareto frontier between coherence and verifiability. The framework places Ijiri's measurement problem within a broader theoretical structure of measurement design under verification constraints, and locates the 1986 and 1990 momentum definitions as regime-conditional choices on a Pareto frontier rather than competing universal claims. Whether the framework helps illuminate other historical or contemporary measurement controversies—OCI recycling, fair-value accounting, the income-versus-wealth debate in international standards—is a question we leave to future work.

A Proofs

A.1 Proof of Lemma 1

We prove attainment for the relaxed problem on $\mathcal{T}_0$, then derive the strict-positivity equivalence.

Attainment on $\mathcal{T}_0$. Under Assumptions 3.1, 3.2, and 5.1, the constraint set $\mathcal{T}_0$ is a closed bounded subset of the finite-dimensional Hilbert space $\mathcal{H}_{\mathrm{op}} = \mathcal{L}(R, M_f)$, where M_f is the finite-dimensional ambient subspace from Assumption 5.1. The condition $\sigma_{\min}(T) \geq c_0$ is closed (it is the complement of an open condition $\sigma_{\min}(T) < c_0$ in the continuous topology of singular values), and the condition $\|T\|_{\mathrm{HS}} = 1$ is closed. Hence $\mathcal{T}_0$ is closed and bounded in a finite-dimensional space, therefore compact.

The functional $\varepsilon(T) = \alpha\|LT\|_{\mathrm{HS}} + \beta\|(I_{\mathcal{L}} - E_V^\sharp)T\|_{\mathrm{HS}}$, viewed as a function of $T \in \mathcal{T}_0$, is continuous: each Hilbert–Schmidt norm is continuous, and L and $E_V^\sharp$ are bounded linear operators. Hence ε is continuous on the compact set $\mathcal{T}_0$, and attains its infimum. This establishes attainment of $\varepsilon^*_{\mathcal{T}_0}$.

Equivalence. Suppose $\mathcal{C}_f \cap \mathcal{V}_f \cap \{T : \sigma_{\min}(T) \geq c_0,\ \|T\|_{\mathrm{HS}} = 1\} \neq \emptyset$, and let T^* be an element of this set. Then $L_f T^* = 0$ (by membership in $\mathcal{C}_f = \ker L_f$) and $E_V^\sharp T^* = T^*$ (by membership in $\mathcal{V}_f$, the fixed-point set of $E_V^\sharp$ in $\mathcal{H}_{\mathrm{op}}$). Hence $\varepsilon_{\mathrm{coh}}(T^*) = 0$ and $\varepsilon_{\mathrm{ver}}(T^*) = 0$, so $\varepsilon^*_{\mathcal{T}_0} \leq \varepsilon(T^*) = 0$. Conversely, if the intersection is empty, then for every $T \in \mathcal{T}_0$ at least one of $L_f T \neq 0$ or $(I_{\mathcal{H}_{\mathrm{op}}} - E_V^\sharp)T \neq 0$ holds, so $\varepsilon(T) > 0$. By the compactness of $\mathcal{T}_0$ and continuity of ε, $\varepsilon^*_{\mathcal{T}_0} = \inf_{\mathcal{T}_0} \varepsilon(T) > 0$.

Inequality with the finite-rank admissible problem. The inclusion $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$ gives $\varepsilon^*_{\mathcal{T}_0} \leq \varepsilon^*_{\mathcal{W},f}$. Hence in particular, if $\varepsilon^*_{\mathcal{T}_0} > 0$, then $\varepsilon^*_{\mathcal{W},f} \geq \varepsilon^*_{\mathcal{T}_0} > 0$, recovering the substantive content of Ijiri's impossibility for $\mathcal{W}_0^f$. □

A.2 Proof of Lemma 2

Recall that $E_V^\sharp : \mathcal{L}(R, H_Y) \to \mathcal{L}(R, H_Y)$ is defined by $E_V^\sharp T := E_V \circ T$.

Idempotency. Since E_V is idempotent ($E_V^2 = E_V$), $(E_V^\sharp)^2 T = E_V \circ E_V \circ T = E_V \circ T = E_V^\sharp T$.

Self-adjointness in the HS inner product. For $S, T \in \mathcal{L}(R, H_Y)$ and any orthonormal basis $\{e_i\}$ of R:

$$\begin{aligned}
\langle E_V^\sharp S, T\rangle_{\mathrm{HS}} &= \sum_i \langle E_V S e_i, T e_i\rangle_{H_Y} \\
&= \sum_i \langle S e_i, E_V T e_i\rangle_{H_Y} \quad \text{(by self-adjointness of } E_V \text{ in } H_Y\text{)} \\
&= \langle S, E_V^\sharp T\rangle_{\mathrm{HS}}.
\end{aligned}$$

Range $= \mathcal{V}$. For $T \in \mathcal{L}(R, H_Y)$, $(E_V^\sharp T)(f) = E_V T f \in M_V$ for all $f \in R$. Hence $\mathrm{Range}(E_V^\sharp) \subseteq \mathcal{V}$. Conversely, for $T \in \mathcal{V}$, $Tf \in M_V$ for all $f \in R$, so $E_V T f = T f$, i.e., $E_V^\sharp T = T$. Thus $\mathcal{V} \subseteq \mathrm{Range}(E_V^\sharp)$, and the reverse inclusion combined with $E_V^\sharp|_{\mathcal{V}} = I|_{\mathcal{V}}$ shows that $\mathrm{Range}(E_V^\sharp) = \mathcal{V}$.

The three properties (idempotency, self-adjointness, fixed range $\mathcal{V}$) characterize $E_V^\sharp$ as the orthogonal projection from $\mathcal{L}(R, H_Y)$ onto $\mathcal{V}$. $\square$

A.3 Proof of Theorem 1

We establish the bound $\varepsilon^*_{\mathcal{T}_0} \geq \sqrt{\lambda_{\min}(A_f)}$ where $A_f = \alpha^2 L_f^* L_f + \beta^2 (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)$ on $\mathcal{H}_{\text{op}} = \mathcal{L}(R, M_f)$.

Step 1: Sub-additive squaring inequality. For non-negative real numbers a, b:

$$(a+b)^2 \geq a^2 + b^2.$$

Hence for $T \in \mathcal{T}_0 \subseteq \mathcal{H}_{\text{op}}$,

$$\varepsilon(T)^2 = (\alpha \|L_f T\|_{\text{HS}} + \beta \|(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T\|_{\text{HS}})^2 \geq \alpha^2 \|L_f T\|^2_{\text{HS}} + \beta^2 \|(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T\|^2_{\text{HS}}.$$

Step 2: Quadratic form representation. Using the HS inner product on $\mathcal{H}_{\text{op}}$:

$$\begin{aligned}
\alpha^2 \|L_f T\|^2_{\text{HS}} &= \alpha^2 \langle L_f T, L_f T\rangle_{\text{HS}} = \alpha^2 \langle L_f^* L_f T, T\rangle_{\text{HS}},\\
\beta^2 \|(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T\|^2_{\text{HS}} &= \beta^2 \langle (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T, (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T\rangle_{\text{HS}}\\
&= \beta^2 \langle (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)^2 T, T\rangle_{\text{HS}}\\
&= \beta^2 \langle (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp) T, T\rangle_{\text{HS}},
\end{aligned}$$

where we used that $I_{\mathcal{H}_{\text{op}}} - E_V^\sharp$ is idempotent and self-adjoint on $\mathcal{H}_{\text{op}}$ (it is the orthogonal projection onto $\mathcal{V}_f^\perp$). Hence

$$\varepsilon(T)^2 \geq \langle T, A_f T\rangle_{\text{HS}},$$

where $A_f := \alpha^2 L_f^* L_f + \beta^2 (I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)$. The operator A_f is a sum of two positive semi-definite self-adjoint operators on the finite-dimensional Hilbert space $\mathcal{H}_{\text{op}}$, hence is itself positive semi-definite and self-adjoint, with discrete spectrum.

Step 3: Rayleigh quotient. For any T on the unit HS-sphere in $\mathcal{H}_{\text{op}}$, $\langle T, A_f T\rangle_{\text{HS}} \geq \lambda_{\min}(A_f)$. Hence

$$\varepsilon(T)^2 \geq \lambda_{\min}(A_f) \quad \text{for all } T \in \mathcal{T}_0,$$

since $\mathcal{T}_0 \subseteq \{T \in \mathcal{H}_{\text{op}} : \|T\|_{\text{HS}} = 1\}$. Taking infimum, $(\varepsilon^*_{\mathcal{T}_0})^2 \geq \lambda_{\min}(A_f)$, hence $\varepsilon^*_{\mathcal{T}_0} \geq \sqrt{\lambda_{\min}(A_f)}$. Since $\mathcal{T}_{\mathcal{W}} \subseteq \mathcal{T}_0$, the bound holds *a fortiori* for the finite-rank admissible problem: $\varepsilon^*_{\mathcal{W},f} \geq \sqrt{\lambda_{\min}(A_f)}$.

Step 4: Positivity equivalence. A_f is positive semi-definite self-adjoint, so $\lambda_{\min}(A_f) \geq 0$. We show $\lambda_{\min}(A_f) > 0 \Leftrightarrow \mathcal{C}_f \cap \mathcal{V}_f = \{0\}$.

If $\mathcal{C}_f \cap \mathcal{V}_f \neq \{0\}$, take $T \in \mathcal{C}_f \cap \mathcal{V}_f$ with $\|T\|_{\text{HS}} = 1$. Then $L_f T = 0$ and $(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T = 0$, so $\langle T, A_f T\rangle_{\text{HS}} = \alpha^2 \|L_f T\|^2_{\text{HS}} + \beta^2 \|(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T\|^2_{\text{HS}} = 0$. Hence $\lambda_{\min}(A_f) = 0$.

Conversely, if $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$, then for every $T \in \mathcal{H}_{\text{op}}$ with $T \neq 0$, at least one of $L_f T \neq 0$ or $(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp)T \neq 0$. Hence $\langle T, A_f T\rangle_{\text{HS}} > 0$ for all nonzero $T \in \mathcal{H}_{\text{op}}$, so A_f is strictly positive definite and $\lambda_{\min}(A_f) > 0$. $\square$

A.4 Proof of Proposition 5.1

We establish the bound $\sqrt{\lambda_{\min}(A_f)} \geq \min\{\alpha\sigma^+_{\min}(L_f), \beta\}\sqrt{1-\cos\theta_F(\mathcal{C}_f, \mathcal{V}_f)}$ under the additional hypothesis $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$.

Step 1: Operator inequality on $\mathcal{C}_f^\perp$ and $\mathcal{V}_f^\perp$. On $\mathcal{C}_f^\perp$ (the orthogonal complement of $\mathcal{C}_f$ in $\mathcal{H}_{\text{op}}$), the operator $L_f^* L_f$ satisfies $\langle T, L_f^* L_f T\rangle_{\text{HS}} = \|L_f T\|^2_{\text{HS}} \geq (\sigma^+_{\min}(L_f))^2\|T\|^2_{\text{HS}}$, since $\sigma^+_{\min}(L_f)$ is the smallest non-zero singular value of L_f and $T \in \mathcal{C}_f^\perp$ has no component in $\ker L_f = \mathcal{C}_f$. Hence

$$\alpha^2 L_f^* L_f \geq \alpha^2(\sigma^+_{\min}(L_f))^2 P_{\mathcal{C}_f^\perp},$$

where $P_{\mathcal{C}_f^\perp}$ is the orthogonal projection onto $\mathcal{C}_f^\perp$. Similarly, $I_{\mathcal{H}_{\text{op}}} - E_V^\sharp = P_{\mathcal{V}_f^\perp}$, so $\beta^2(I_{\mathcal{H}_{\text{op}}} - E_V^\sharp) = \beta^2 P_{\mathcal{V}_f^\perp}$.

Step 2: Lower-bound combination. Adding,

$$A_f = \alpha^2 L_f^* L_f + \beta^2 P_{\mathcal{V}_f^\perp} \geq \alpha^2(\sigma^+_{\min}(L_f))^2 P_{\mathcal{C}_f^\perp} + \beta^2 P_{\mathcal{V}_f^\perp} \geq \min\{\alpha^2(\sigma^+_{\min}(L_f))^2, \beta^2\}(P_{\mathcal{C}_f^\perp} + P_{\mathcal{V}_f^\perp}).$$

Step 3: Quadratic form on the unit sphere. For unit $T \in \mathcal{H}_{\text{op}}$:

$$\langle T, (P_{\mathcal{C}_f^\perp} + P_{\mathcal{V}_f^\perp})T\rangle_{\text{HS}} = \|P_{\mathcal{C}_f^\perp} T\|^2 + \|P_{\mathcal{V}_f^\perp} T\|^2.$$

We minimize this over the unit sphere. When $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$, the minimum is achieved at unit vectors that lie "close" to both $\mathcal{C}_f$ and $\mathcal{V}_f$ but cannot lie in their intersection. By the Friedrichs angle characterization,

$$\min_{\|T\|=1}(\|P_{\mathcal{C}_f^\perp} T\|^2 + \|P_{\mathcal{V}_f^\perp} T\|^2) = 1 - \cos\theta_F(\mathcal{C}_f, \mathcal{V}_f).$$

The latter equality follows from the fact that the minimum of $\|P_{\mathcal{C}_f^\perp} T\|^2 + \|P_{\mathcal{V}_f^\perp} T\|^2 = 2 - \|P_{\mathcal{C}_f} T\|^2 - \|P_{\mathcal{V}_f} T\|^2$ on the unit sphere equals $2 - \max_{\|T\|=1}(\|P_{\mathcal{C}_f} T\|^2 + \|P_{\mathcal{V}_f} T\|^2)$, and the latter maximum is $1 + \cos\theta_F(\mathcal{C}_f, \mathcal{V}_f)$ when $\mathcal{C}_f \cap \mathcal{V}_f = \{0\}$.

Step 4: Combining.

$$\lambda_{\min}(A_f) \geq \min\{\alpha^2(\sigma^+_{\min}(L_f))^2, \beta^2\} \cdot (1 - \cos\theta_F(\mathcal{C}_f, \mathcal{V}_f)).$$

Taking square roots,

$$\sqrt{\lambda_{\min}(A_f)} \geq \min\{\alpha\sigma^+_{\min}(L_f), \beta\} \cdot \sqrt{1 - \cos\theta_F(\mathcal{C}_f, \mathcal{V}_f)},$$

as claimed. □

A.5 Proof of Theorem 2

We verify each of the five parts in turn.

Part (1): Admissibility. We verify that $W^* = E_V \circ W^*_{\text{coh}}$ satisfies (W1)–(W4) of Definition 3.3.

(W1) Linearity: W^* is the composition of two linear operators.

(W2) Boundedness: $\|W^*\|_{\mathrm{op}} \le \|E_V\|_{\mathrm{op}} \cdot \|W^*_{\mathrm{coh}}\|_{\mathrm{op}} = 1 \cdot \|W^*_{\mathrm{coh}}\|_{\mathrm{op}} < \infty$.

(W3) Positivity preservation: $W^*_{\mathrm{coh}} \in \mathcal{W}$ is admissible, hence positivity-preserving. E_V is conditional expectation, which preserves positivity. Composition of positivity-preserving maps is positivity-preserving.

(W4) Constant preservation: $W^*[\mathbf{1}_Z] = E_V(W^*_{\mathrm{coh}}[\mathbf{1}_Z]) = E_V[\mathbf{1}_Y] = \mathbf{1}_Y$, where the last equality uses property (P5) of the conditional expectation.

Part (2): Zero verifiability defect. The range of W^* is $E_V(\mathrm{Range}(W^*_{\mathrm{coh}})) \subseteq M_V$. Hence $(I_Y - E_V)W^* = 0$, so $\varepsilon_{\mathrm{ver}}(W^*) = \|(I_Y - E_V)W^* P_R\|_{\mathrm{HS}} = 0$.

Part (3): Closed-form coherence defect. We compute:

$$\begin{aligned}(\delta_Y W^* - W^* \delta_Z) P_R &= (\delta_Y E_V W^*_{\mathrm{coh}} - E_V W^*_{\mathrm{coh}} \delta_Z) P_R \\ &= \delta_Y E_V W^*_{\mathrm{coh}} P_R - E_V W^*_{\mathrm{coh}} \delta_Z P_R.\end{aligned}$$

By the δ_Z-invariance of R (Assumption 3.2), $\delta_Z P_R = P_R \delta_Z P_R$, so

$$W^*_{\mathrm{coh}} \delta_Z P_R = W^*_{\mathrm{coh}} P_R \delta_Z P_R = W^*_{\mathrm{coh}} \delta_R P_R,$$

where we used that $W^*_{\mathrm{coh}} P_R$ acts trivially when followed by P_R on its right (since $P_R^2 = P_R$). By the coherence of W^*_{coh} on R (membership in $\mathcal{C}_f$), $W^*_{\mathrm{coh}} \delta_R = \delta_Y W^*_{\mathrm{coh}}$ on R. Hence

$$W^*_{\mathrm{coh}} \delta_Z P_R = \delta_Y W^*_{\mathrm{coh}} P_R.$$

Substituting back:

$$\begin{aligned}(\delta_Y W^* - W^* \delta_Z) P_R &= \delta_Y E_V W^*_{\mathrm{coh}} P_R - E_V \delta_Y W^*_{\mathrm{coh}} P_R \\ &= (\delta_Y E_V - E_V \delta_Y) W^*_{\mathrm{coh}} P_R \\ &= [\delta_Y, E_V]\, W^*_{\mathrm{coh}}\, P_R.\end{aligned}$$

Taking Hilbert–Schmidt norm,

$$\varepsilon_{\mathrm{coh}}(W^*) = \|[\delta_Y, E_V]\, W^*_{\mathrm{coh}}\, P_R\|_{\mathrm{HS}}.$$

Part (4): Canonical-ness as projection. Let $T^*_{\mathrm{coh}} := W^*_{\mathrm{coh}}|_R \in \mathcal{H}_{\mathrm{op}}$. We show that $T^* := E_V \circ T^*_{\mathrm{coh}} = \arg\min_{T \in \mathcal{V}_f} \|T - T^*_{\mathrm{coh}}\|_{\mathrm{HS}}$. By Lemma 2, $E_V^\sharp$ restricts to the orthogonal projection from $\mathcal{H}_{\mathrm{op}}$ onto $\mathcal{V}_f$, so $E_V^\sharp(T^*_{\mathrm{coh}}) = E_V \circ T^*_{\mathrm{coh}} = T^*$ is the (unique) HS-orthogonal projection of T^*_{coh} onto $\mathcal{V}_f$.

Part (5): Determinacy modulo coherent selection. Once W^*_{coh} is fixed (Assumption 6.1), the composition $W^* = E_V \circ W^*_{\mathrm{coh}}$ is determined. We illustrate one source of non-uniqueness in the coherent selection itself. On $\ker \delta_R \cap R$, the coherence equation $\delta_Y T f = T \delta_R f = 0$ requires $Tf \in \ker \delta_Y$, which is a non-trivial subspace whenever δ_Y is not injective. Different choices of $T|_{\ker \delta_R \cap R}$ within $\ker \delta_Y$ yield different elements of $\mathcal{C}_f$. The HS-norm minimization in the definition of W^*_{coh} resolves part of this ambiguity by selecting an HS-norm minimizer; multiple HS-norm minimizers can coexist (e.g., when δ_R has repeated eigenvalues or δ_Y has additional symmetries), so the kernel ambiguity is one identifiable source of non-uniqueness but not necessarily the only one. $\square$

A.6 Proof of Proposition 7.1

From (3),

$$\varepsilon(W_{1990}) - \varepsilon(W_{1986}) = \tfrac{\alpha}{\sqrt{3}}(\sigma_{\text{inc}} - \sigma_{\text{stock}}) + \beta(\sqrt{1-\rho^2} - \rho) = \tfrac{\alpha\Delta}{\sqrt{3}} + \beta(\sqrt{1-\rho^2} - \rho).$$

Setting this to zero yields

$$\sqrt{1-\rho^2} - \rho = -\frac{\alpha\Delta}{\beta\sqrt{3}} = -k.$$

Range of the left-hand side. The function $g(\rho) := \sqrt{1-\rho^2} - \rho$ is continuous on $[0,1]$, with $g(0) = 1$ and $g(1) = -1$. Computing the derivative, $g'(\rho) = -\rho/\sqrt{1-\rho^2} - 1 < 0$ for $\rho \in [0,1)$, so g is strictly decreasing. Hence g maps $[0,1]$ bijectively onto $[-1,1]$. The equation $g(\rho) = -k$ therefore has a unique solution $\rho \in [0,1]$ if and only if $-k \in [-1,1]$, equivalently $k \in [-1,1]$.

Closed-form solution. Let $\rho = \cos\phi$ for $\phi \in [0, \pi/2]$, so $\sqrt{1-\rho^2} = \sin\phi$. The equation becomes $\sin\phi - \cos\phi = -k$, equivalently $\sqrt{2}\sin(\phi - \pi/4) = -k$, hence

$$\sin(\phi - \pi/4) = -\frac{k}{\sqrt{2}}.$$

For $\phi \in [0, \pi/2]$, the argument $\phi - \pi/4$ ranges over $[-\pi/4, \pi/4]$, on which sin takes values in $[-1/\sqrt{2}, 1/\sqrt{2}]$. The equation has a solution in this range if and only if $|-k/\sqrt{2}| \le 1/\sqrt{2}$, i.e., $|k| \le 1$, consistent with the range analysis above. The solution is $\phi = \pi/4 - \arcsin(k/\sqrt{2})$, giving $\rho = \cos\phi = \cos(\pi/4 - \arcsin(k/\sqrt{2}))$. By the cosine angle-sum formula,

$$\begin{aligned}
\rho^* &= \cos(\pi/4)\cos(\arcsin(k/\sqrt{2})) + \sin(\pi/4)\sin(\arcsin(k/\sqrt{2})) \\
&= \frac{1}{\sqrt{2}}\sqrt{1-k^2/2} + \frac{1}{\sqrt{2}} \cdot \frac{k}{\sqrt{2}} \\
&= \frac{\sqrt{2-k^2}}{2} + \frac{k}{2} = \frac{k + \sqrt{2-k^2}}{2}.
\end{aligned}$$

This proves part (a).

For (b) and (c), monotonicity follows from

$$\partial_\rho[\varepsilon(W_{1990}) - \varepsilon(W_{1986})] = \beta\,\partial_\rho(\sqrt{1-\rho^2} - \rho) = -\beta\left(\frac{\rho}{\sqrt{1-\rho^2}} + 1\right) < 0$$

for $\rho \in [0,1)$. Hence the difference is strictly decreasing in ρ, so $\varepsilon(W_{1990}) < \varepsilon(W_{1986})$ for $\rho > \rho^*$ and the reverse for $\rho < \rho^*$.

(d) At $\Delta = 0$, $k = 0$, so $\rho^*(0) = (0 + \sqrt{2})/2 = 1/\sqrt{2}$.

(e) For $|k| > 1$, the range analysis shows that $g(\rho) = -k$ has no solution in $[0,1]$. Since g is continuous and the inequality $g(\rho) > -k$ (or $< -k$) holds throughout $[0,1]$, the difference $\varepsilon(W_{1990}) - \varepsilon(W_{1986})$ does not change sign on $\rho \in [0,1]$, and one map dominates the other across all admissible ρ. $\square$

B Numerical Illustration and Generalization

B.1 A Numerical Example

We illustrate the closed-form expressions of (3) and the threshold ρ^* of Proposition 7.1 with a specific numerical specification.

Spectral data. Take the recirculation eigenvalues

$$\lambda_A = -1.8, \qquad \lambda_D = 0.4, \qquad \lambda_O = -0.3,$$

representing strong reversal of accruals (most of the value reverses next period), partial recognition of deferred items (slight positive growth), and slow recycling of OCI items (mild negative trend). Take the wealth-side dynamics

$$\mu_{\text{inc}} = -0.2, \qquad \mu_{\text{stock}} = -1.0,$$

representing modest negative income trend and stronger negative stock-side dynamics.

Coherence defects. Direct computation yields

$$\begin{aligned}
\sigma_{\text{inc}}^2 &= (\mu_{\text{inc}} - \lambda_A)^2 + (\mu_{\text{inc}} - \lambda_D)^2 + (\mu_{\text{inc}} - \lambda_O)^2 \\
&= (-0.2 + 1.8)^2 + (-0.2 - 0.4)^2 + (-0.2 + 0.3)^2 \\
&= 2.56 + 0.36 + 0.01 = 2.93, \\
\sigma_{\text{inc}} &\approx 1.712.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sigma_{\text{stock}}^2 &= (-1.0 + 1.8)^2 + (-1.0 - 0.4)^2 + (-1.0 + 0.3)^2 \\
&= 0.64 + 1.96 + 0.49 = 3.09, \\
\sigma_{\text{stock}} &\approx 1.758.
\end{aligned}$$

The coherence asymmetry is $\Delta = \sigma_{\text{inc}} - \sigma_{\text{stock}} \approx -0.046$: in this specification, income-side dynamics produce slightly smaller coherence defect than stock-side dynamics.

Threshold value. With symmetric institutional weights $\alpha = \beta = 1$:

$$k = \frac{\alpha\Delta}{\beta\sqrt{3}} = \frac{-0.046}{1.732} \approx -0.027.$$

Hence

$$\rho^*(k) = \frac{k + \sqrt{2 - k^2}}{2} \approx \frac{-0.027 + \sqrt{1.999}}{2} \approx \frac{-0.027 + 1.4138}{2} \approx 0.694.$$

The threshold lies slightly below the spectrally symmetric value $1/\sqrt{2} \approx 0.707$, reflecting the small income-favoring asymmetry.

Tabulated comparison. For specific values of ρ, with $\alpha/\sqrt{3}\cdot\sigma_{\text{stock}} \approx 1.015$ and $\alpha/\sqrt{3}\cdot\sigma_{\text{inc}} \approx 0.989$:

ρ	$\varepsilon(W_{1986})$	$\varepsilon(W_{1990})$	Smaller
0.10	$1.015 + 0.100 = 1.115$	$0.989 + 0.995 = 1.984$	W_{1986}
0.30	$1.015 + 0.300 = 1.315$	$0.989 + 0.954 = 1.943$	W_{1986}
0.50	$1.015 + 0.500 = 1.515$	$0.989 + 0.866 = 1.855$	W_{1986}
0.694	$1.015 + 0.694 = 1.709$	$0.989 + 0.720 = 1.709$	(tied)
0.71	$1.015 + 0.710 = 1.725$	$0.989 + 0.704 = 1.693$	W_{1990}
0.85	$1.015 + 0.850 = 1.865$	$0.989 + 0.527 = 1.516$	W_{1990}
0.95	$1.015 + 0.950 = 1.965$	$0.989 + 0.312 = 1.301$	W_{1990}

The table confirms the behavior described by Proposition 7.1: W_{1986} has smaller gap when ρ is below $\rho^* \approx 0.694$, W_{1990} when ρ is above, and the two are equal at the threshold.

B.2 Generalizations Beyond the Homogeneous Toy Model

The closed-form decomposition (3) relies on the homogeneity of wealth-side dynamics across components ($\mu_{\text{inc}}, \mu_{\text{stock}}$ independent of k) and the uniformity of the regime parameter ρ across components.

Heterogeneous wealth dynamics. If $(\mu^k_{\text{inc}}, \mu^k_{\text{stock}})$ varies with the recirculation component k, the spectral quantities $\sigma_{\text{inc}}, \sigma_{\text{stock}}$ generalize to

$$\sigma^2_{\text{inc}} = \sum_k (\mu^k_{\text{inc}} - \lambda_k)^2, \qquad \sigma^2_{\text{stock}} = \sum_k (\mu^k_{\text{stock}} - \lambda_k)^2.$$

The threshold ρ^* formula generalizes accordingly. The qualitative regime-conditional structure of Proposition 7.1 is preserved.

Heterogeneous verification regime. If different components of R are subject to different verification regimes (ρ_k varying with k), the verifiability defects (under the unit-HS-norm normalization of Section 7.3) generalize to

$$\varepsilon_{\text{ver}}(W_{1986}) = \tfrac{1}{\sqrt{3}}\sqrt{\sum_k \rho_k^2}, \qquad \varepsilon_{\text{ver}}(W_{1990}) = \tfrac{1}{\sqrt{3}}\sqrt{\sum_k (1-\rho_k^2)}.$$

The threshold structure is replaced by a higher-dimensional set of regime-conditional inequalities.

Higher-dimensional recirculation subspaces. For $\dim R > 3$, the framework extends naturally with the same coherence and verifiability defect formulas, with sums running over the additional components. The closed-form ordering structure of Proposition 7.1 continues to hold, with appropriate generalization of the spectral quantities.

Infinite-dimensional generalization. For R infinite-dimensional, the framework requires more care. The compactness argument in the proof of Lemma 1 fails, and attainment of the infimum is not automatic. The closed-form expressions of (3) extend formally if the relevant series converge (which is the case when $\sum_k (\mu - \lambda_k)^2 < \infty$ for $\mu = \mu_{\text{inc}}, \mu_{\text{stock}}$). The Friedrichs-angle interpretation of Proposition 5.1 also extends, but the computation of the angle becomes substantially more involved. We leave the infinite-dimensional case as future work.

References

Basu, S. (1997). The conservatism principle and the asymmetric timeliness of earnings. *Journal of Accounting and Economics* 24(1): 3–37.

Black, D. E. (2016). Other comprehensive income: A review and directions for future research. *Accounting & Finance* 56(1): 9–45.

Bradbury, M. E. (2016). Discussion of 'Other comprehensive income: A review and directions for future research'. *Accounting & Finance* 56(1): 47–58.

Bushman, R. M. and A. J. Smith (2001). Financial accounting information and corporate governance. *Journal of Accounting and Economics* 32(1–3): 237–333.

Chambers, R. J. (1966). *Accounting, Evaluation and Economic Behavior*. Englewood Cliffs, NJ: Prentice-Hall.

Christensen, J. A. and J. S. Demski (2003). *Accounting Theory: An Information Content Perspective*. New York: McGraw-Hill/Irwin.

Christensen, P. O. and G. A. Feltham (2003). *Economics of Accounting, Volume I: Information in Markets*. Boston: Springer.

Demski, J. S. (1973). The general impossibility of normative accounting standards. *The Accounting Review* 48(4): 718–723.

Demski, J. S. (1980). *Information Analysis*. 2nd ed. Reading, MA: Addison-Wesley.

Demski, J. S., S. A. FitzGerald, Y. Ijiri, Y. Ijiri, and H. Lin (2006). Quantum information and accounting information: Their salient features and conceptual applications. *Journal of Accounting and Public Policy* 25(4): 435–464.

Demski, J. S., S. A. FitzGerald, Y. Ijiri, Y. Ijiri, and H. Lin (2009). Quantum information and accounting information: Exploring conceptual applications of topology. *Journal of Accounting and Public Policy* 28(2): 133–147.

Friedrichs, K. (1937). On certain inequalities and characteristic value problems for analytic functions and for functions of two variables. *Transactions of the American Mathematical Society* 41(3): 321–364.

Ijiri, Y. (1975). *Theory of Accounting Measurement*. Studies in Accounting Research No. 10. Sarasota, FL: American Accounting Association.

Ijiri, Y. (1986). A framework for triple-entry bookkeeping. *The Accounting Review* 61(4): 745–759.

Ijiri, Y. (1989). *Momentum Accounting and Triple-Entry Bookkeeping: Exploring the Dynamic Structure of Accounting Measurements*. Studies in Accounting Research No. 31. Sarasota, FL: American Accounting Association.

Ijiri, Y. (1990). *'Risoku Kaikei' Nyumon: Kigyo Seicho e no Shin Gyoseki Hyoka Shisutemu* [Introduction to 'Income-Velocity Accounting': A New Performance Evaluation System for Corporate Growth]. Tokyo: Nihon Keizai Shimbunsha.

Lawrence, A., S. Siriviriyakul, and R. G. Sloan (2016). Who's the fairest of them all? Evidence from closed-end funds. *The Accounting Review* 91(1): 207–227.

Linsmeier, T. J. (2016). Revised model for presentation in statement(s) of financial performance: Potential implications for measurement in the conceptual framework. *Accounting Horizons* 30(4): 485–498.

Magnan, M., A. Menini, and A. Parbonetti (2015). Fair value accounting: Information or confusion for financial markets? *Review of Accounting Studies* 20(1): 559–591.

Mattessich, R. (1957). Towards a general and axiomatic foundation of accountancy: With an introduction to the matrix formulation of accounting systems. *Accounting Research* 8(4): 328–355.

Mattessich, R. (1964). *Accounting and Analytical Methods*. Homewood, IL: R. D. Irwin.

Mattessich, R. (1995). *Critique of Accounting: Examination of the Foundations and Normative Structure of an Applied Discipline*. Westport, CT: Quorum Books.

Okada, Y. and Y. Koike (2021). Momentum Accounting kenkyu no shiteki tenkai to shorai no hatten hoko [Historical development of Momentum Accounting research and future directions]. *Journal of Japanese Association for Bookkeeping Studies* 4(1): 12–25.

Song, C. J., W. B. Thomas, and H. Yi (2010). Value relevance of FAS No. 157 fair value hierarchy information and the impact of corporate governance mechanisms. *The Accounting Review* 85(4): 1375–1410.

Suzuki, T. (2026a). Path-dependent accounting measurement and the minimal structure of triple-entry bookkeeping: Residual closure, verifiability, and the characterization of existence conditions. Working paper.

Suzuki, T. (2026b). Wealth-based and income-based momentum in accounting measurement: A noncommutative reconstruction of Ijiri. Working paper.

Watts, R. L. (2003a). Conservatism in accounting part I: Explanations and implications. *Accounting Horizons* 17(3): 207–221.

Watts, R. L. (2003b). Conservatism in accounting part II: Evidence and research opportunities. *Accounting Horizons* 17(4): 287–301.

Whittington, G. (2008). Fair value and the IASB/FASB conceptual framework project: An alternative view. *Abacus* 44(2): 139–168.