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# A Matching Model with Individual Bounded Money Holdings and the Nash Bargaining Rule.

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**Abstract**

This paper analyzes a Trejos-Wright model with individual money holdings in the set $\{0, 1, 2\}$. Existing papers investigate the model which assumes a take-it-or-leave-it offer by potential buyers. We assume the Nash bargaining rule and clarify the distinction of the conclusions. It is well known that three kind of monetary steady states equilibria exist: (1) pure-strategy full-support steady states, (2) mixed-strategy full support steady states, and (3) non-full-support steady states. In the take-it-or-leave-it offer, full-support steady state equilibria exist if pure-strategy full-support steady state equilibria does not. We show some conjectures by using a numerical analysis. we show that existence of steady state equilibria (1), (2) and (3) depend on a bargaining power.

**Keywords** : Money, search, Decentralized Trade, Non-degenerated distribution.
**JEL classification** : C60, E40, E50

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# 1 Introduction

Shi (1995) and Trejos and Wright (1995) introduce the random matching models with divisible goods, an indivisible fiat money, and individual money holdings limited to either zero or one unit. In the same models, Zhu (2003) permits individuals to hold a bounded but non-negative integer units of the money[1]. In addition, trading pairs in single-coincidence meetings make a bargain by take-it-or-leave-it offers by potential buyers. For this setting, Zhu (2003) provides sufficient conditions for existence of a full-support monetary steady state equilibria with a strictly increasing and strictly concave value function, and this result also implies the existence of non-full-support steady state equilibria. These steady states equilibria are classified as the following types: (1) pure-strategy full-support steady states, (2) mixed-strategy full support steady states, and (3) non-full-support steady states. Assumption of take-it-or-leave-it offers plays an important role for Zhu's study. This assumption assures concavity properties of value function. For other bargaining rules, the properties are not preserved, therefore the analytical study is difficult. For this reason, we can not use Zhu's technique if other bargaining rules are assumed.

As a resemble analysis, Molico (2006) studies the model with divisible money, divisible goods, unbounded money holdings, and the Nash bargaining rule numerically. Actually, he approximates the economy with the unbounded money holdings, setting the upper bound on money holdings at 100 times the average money holdings. However, in his numerical study, steady states equilibria of three types are not discuss.

In general, choice of the bargaining rules affects value of money in steady states equilibria. For instance, Rocheteau and Waller (2005) analyzes the effects. Their analysis is based on Lagos and Wright (2005). In Lagos and Wright's economy, households participate in a decentralized "day market" and a centralized "night market" every period. Households' money holdings at the end of the day market diverge depending on the trade status in the day market. The households, however, adjust the money holdings at the night market and the money holding distribution degenerates at the beginning of the next period. In this framework, Rocheteau and Waller (2005) investigate three bargaining rules: the egalitarian rule, the Kalai-Smorodinsky rule, and the Nash rule. For instance, efficient trades do not realize on monetary equilibria when the Nash rule is assumed. On the other hand, in the case of the egalitarian rule, efficient trades can realize. However, since their study is based on the economy in which money holdings distribution is degenerated, they can not study the effects on the distribution.

Our study clarifies the effects on the money distribution, welfare, GDP, and the steady states types by choice of the Nash bargaining rule. For that, this paper studies the Zhu's economy with the Nash bargaining rule, but money holdings are limited in the set $\{0, 1, 2\}$[2]. We analyze the effects obtained by changing of seller's bargaining power. Finally, we show some conjectures.

[1] Taber and Wallace (1999) analyzes the Zhu's economy with commodity money. Individuals gain a direct utility payoff from the money.

[2] Huang and Igarashi (2015) studies the Zhu's economy with money holdings limited in the set $\{0, 1, 2\}$ and the bargaining rule by take-it-or-leave-it offers. They investigate stabilities of full-support monetary steady state equilibria and non-full-support steady state equilibria.

## 2 The Model

We construct a Trejos-Wright model with multi-unit and the upper bound of money holdings. We consider a dynamic economy with time index $t \in \{0, 1, 2, \cdots\}$. There is one unit of continuous, homogeneous, and infinitely lived household. Households trade heterogeneous special good and fiat money. Special good is non-durable and divisible, and money is indivisible. A seller can produce the good. A buyer can consume the real good if it is not produced by themselves so that exchange is necessary. The exchange occurs as the results of bargaining by randomly matched pairs. We assume money holdings of matched pairs are observable here, and assume random matching without double coincidence. Probability to be a buyer (a seller) in a single coincidence pair is $\alpha < 1/2$, and $1 - 2\alpha$ is probability of no-coincidence.

Household's utility function is

$$\sum_{t=0}^{\infty} \beta^t (U(z_t) - C(z'_t)), \tag{1}$$

where $z_t$ is consumption at $t$, and $z'_t$ is production at $t$. $U, C$ are strictly increasing, twice continuously differentiable, satisfying $U(0) = C(0) = 0$, $U' > 0$, $C' > 0$, $U'' < 0$, $C'' > 0$, $\lim_{z \to 0} U'(z)/C'(z) = \infty$, and there exists a non-trivial efficient production level $\tilde{z}$ such that $U'(\tilde{z}) = C'(\tilde{z})$. The discount factor $\beta$ is between 0 and 1.

Each household's possible set of money holding is $x \in X \equiv \{0, 1, .., \bar{x}\}$, where the upper bound of money holdings is represented by $\bar{x} \in \mathbb{N}$. Total money supply (nominal) is $M$. Let $F(x)$ denote the population of households holding $x$ unit of money. Since total population is 1, we must have

$$\sum_{x=0}^{\bar{x}} F(x) = 1, \tag{2}$$

$$\sum_{x=0}^{\bar{x}} xF(x) = M. \tag{3}$$

$F$ is in $\bar{x} + 1$ dimensional simplex. Given $M$, degree of freedom of $F$ is $\bar{x} - 1$. Thus $F$ is represented by $(F(0), F(1), ..., F(\bar{x} - 2)) \in \mathbb{R}_+^{\bar{x}-1}$

Consider a single-coincidence pair. Let $x$ and $x'$ respectively denote money holdings of the buyer and the seller. They conduct a bargain over $(y, z)$, where $y = Y(x, x')$ is money spending, and $z = Z(x, x')$ is production. It satisfies $Y(x, x') \leq x$ and $Y(x, x') + x' \leq \bar{x}$. The space of $Y$ is finite. Suppose $\bar{x} = 2$, there are 24 possible cases of the matrix $Y$, where $Y$ can be represented by $3 \times 3$ matrix, and the number of rows and the number of columns respectively shows the buyer's money holdings $(x)$ and the seller's money holdings $(x')$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 \text{ or } 1 & 0 \text{ or } 1 & 0 \\ 0 \text{ or } 1 \text{ or } 2 & 0 \text{ or } 1 & 0 \end{pmatrix}, \tag{4}$$

Let $V(x)$ denote the value of a household with $x$. Value function is $V(x) : X \to \mathbb{R}$. Given $V$, we can solve $Y, Z$. Nash bargaining solution are represented as

$$(Y(x,x'), Z(x,x')) \in \text{argmax}_{(y,z)} (G^b)^\gamma \cdot (G^s)^{1-\gamma} \quad \text{s.t. } y \leq \min(x, \bar{x} - x'), \tag{5}$$

where $\gamma$ is buyer's bargaining power, and

$$G^b(y,z;x,x') = U(z) + \beta(V(x-y) - V(x)), \tag{6}$$
$$G^s(y,z;x',x) = -C(z) + \beta(V(x'+y) - V(x')). \tag{7}$$

$Y$ and $F$ determines the transition probability matrix of $F$. Not like the Markov process, the transition matrix depends on $F$. If $\bar{x}$ is 2 and matrix $Y(x,x')$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \tag{8}$$

then

$$F_{+1} = (F(0), F(1), F(2)) \cdot$$
$$\begin{pmatrix} (1-2\alpha) + \alpha + \alpha F(0) & \alpha(F(1)+F(2)) & 0 \\ \alpha(F(0)+F(1)) & (1-2\alpha) + \alpha(F(0)+F(2)) & \alpha(F(1)+F(2)) \\ 0 & \alpha(F(0)+F(1)) & (1-2\alpha)+\alpha+\alpha F(2) \end{pmatrix}. \tag{9}$$

Given $Y$, this is a mapping from $F$ to $F$. If $F$ converges to $F^*$ by repeating the mapping, $F^*$ is the stationary distribution of money holdings.

Given $V, F, Z$ and $Y$, $V(x)$ satisfies,

$$\begin{aligned} V(x) &= \alpha(\sum F(x')(U(Z(x,x')) + \beta V(x - Y(x,x'))) \\ &+ \alpha(\sum F(x')(-C(Z(x',x) + \beta V(x + Y(x',x))) + (1-2\alpha)\beta V(x) \end{aligned} \tag{10}$$

Solve a SSE given $M$:

$$V \to (Y, Z), \tag{11}$$
$$(F, Y) \to F', \quad \Rightarrow \quad Y \to F, \tag{12}$$

and

$$(V, Z, Y, F) \to V'. \tag{13}$$

The whole process is summarized by a mapping from $V$ to $V'$. If $V$ converges to $V^*$ by repeating the mapping, $V^*$ is SSE. of money holdings. In this paper we see monetary equilibria only, that is, $V$ is larger than zero.

**Definition 1** *Given $F_0$ and $M$, the steady state equilibria is $(F^*, V^*)$ that satisfies the Nash solution (5), the (general) law of motion (9), the value function (10).*

# 3 Monetary Steady states

In this section we study three monetary steady states: (A) pure-strategy full-support steady states, (B) mixed-strategy full support steady states, and (C) non-full-support steady states.

Case (A).

Consider $(2,0)$-meetings in which a buyer holds 2 units money and a seller 0. Let $W(=(G^b)^\gamma(G^s)^{1-\gamma})$ denote the Nash product. If the buyer pays 2 units in the $(2,0)$-meetings, the Nash product is represented by $W(2,0|y=2)$. By using the notation, we define the probability $l$. Let $l$ be the probability that a buyer pays 1 unit in $(2,0)$-meetings. Let $d$ be the population holding 1 unit.

**Definition 2** *If $W(2,0|y=1)$ is larger than $W(2,0|y=2)$ in a steady state, this implies $l=1$ and $d>0$. Then, the steady state is called a pure-strategy full-support steady state. $Y$ matrix is represented by:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{14}$$

Case (B).

**Definition 3** *If $W(2,0|y=1)$ is equal to $W(2,0|y=2)$ in a steady state, this implies $0<l<1$ and $d>0$. Then, the steady state is called a mixed-strategy full-support steady state. $Y$ matrix is represented by:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 \text{ } or \text{ } 2 & 1 & 0 \end{pmatrix}. \tag{15}$$

Case (C).

**Definition 4** *If $W(2,0|y=1)$ is smaller than $W(2,0|y=2)$ in a steady state, this implies $l=0$ and $d=0$. Then, the steady state is called a non-full-support steady state. $Y$ matrix is represented by:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 \text{ } or \text{ } 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}. \tag{16}$$

# 4 Numerical analysis

In this section, we investigate the economy numerically. First we specify preference and normalize it.

## 4.1 Specification of the Preference

We assume that the felicity functions, $U$ and $C$, are power: $U(z) = U_0 z^u$, and $C(z) = C_0 z^c$, where the four parameters, $U_0, u, C_0$, and $c$, satisfy the following inequality conditions: $U_0 > 0, \quad 0 < u < 1, \quad C_0 > 0, \quad$ and $c \geq 1$.

We apply two normalizations to the specifications. We set the optimal production level, $\tilde{z}$, to be unity. This normalization implies,

$$U_0 \cdot u = C_0 \cdot c.$$

We also set $U(1) - C(1)$ to be unity. This normalization implies,

$$U_0 - C_0 = 1.$$

We set these parameters as $U_0 = 1.99$, $C_0 = 0.99$, and $u = 0.5$. We set $c = 1.0$ to follow the preceding study.

Let us define money supply relative to a maximum by $m = M/\bar{x}$, where $M$ is the total money supply (nominal) and $\bar{x}$ is the upper bound of money holdings. Thus, $m$ is in $(0, 1)$.

## 4.2 Existence of monetary equilibria

We examine existence of monetary equilibria in this subsection. We classify the economy as the the following 8 categories, namely, Category 1 $\sim$ Category 8. Table 1 represents combination of the equilibria types. NF, PF, and MF show non-full-support steady states, pure-strategy full-support steady states, and mixed-strategy full support steady states respectively. For instance, see row of Category 7. PF is 1, and MF is $-1$. 1 indicates that a monetary equilibrium exists. On the other hand, $-1$ shows that our numerical program does not find a monetary equilibrium.

Table 1: Types of equilibria

| Category | NF | PF | MF |
|---|---|---|---|
| 1 | −1 | −1 | −1 |
| 2 | −1 | −1 | 1 |
| 3 | −1 | 1 | −1 |
| 4 | −1 | 1 | 1 |
| 5 | 1 | −1 | −1 |
| 6 | 1 | −1 | 1 |
| 7 | 1 | 1 | −1 |
| 8 | 1 | 1 | 1 |

Table 2, 3 and 4 show the cases of the various bargaining power, $\gamma$. These tables show the following suggestion. First, as the buyer's bargaining power increases, pure-strategy full-support steady states likely exist. Second, there exists the possibility of co-existence of pure-strategy full-support steady states and mixed-strategy full support steady states. In Zhu (2003), full-support steady state equilibria exist if and only if pure-strategy full-support steady state equilibria does not, that is, co-existence is impossible. In our economy, the co-existence (Category 8) likely emerges when the relative money supply, $m$, and the seller's bargaining power, $1-\gamma$, is large.

Table 2: Existence in the case of $\gamma$=0.3

| $\beta$ \ m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 7 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 5 |
| 0.5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 5 | 8 |
| 0.3 | 6 | 6 | 6 | 6 | 6 | 5 | 5 | 5 | 8 |
| 0.1 | 5 | 5 | 5 | 5 | 5 | 6 | 8 | 8 | 8 |

Table 3: Existence in the case of $\gamma$=0.6

| $\beta$ \ m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 6 |
| 0.5 | 7 | 7 | 7 | 7 | 7 | 7 | 6 | 6 | 6 |
| 0.3 | 7 | 7 | 7 | 7 | 7 | 7 | 6 | 6 | 6 |
| 0.1 | 7 | 7 | 6 | 6 | 5 | 6 | 6 | 6 | 8 |

Table 4: Existence in the case of $\gamma$=1

| $\beta$ \ m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| 0.5 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 6 |
| 0.3 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 5 | 6 |
| 0.1 | 7 | 7 | 7 | 7 | 7 | 7 | 5 | 6 | 6 |

### 4.3 The effects on welfare and GDP

Finally, we examine the effects on welfare and GDP. Table 5 shows the relationship between the relative money supply and welfare or GDP. When the money supply is 0.3 given $\gamma$ and $\beta$, welfare is the maximum. On the other hand, GDP is the maximum at 0.2. We can not find a monetary equilibrium at 0.6.

Table 6 shows the relationship between the bargaining power, given the money supply corresponding to the maximum welfare obtained from Table 5, and welfare or GDP. Figure 1-2 show the following suggestion. First, the relative money supply corresponding to the maximum GDP is smaller than that of the maximum welfare. Second, as the buyer's bargaining power increases, GDP increases. Added to this, Figure 2 indicate that welfare and the bargaining power have a single-peaked relationship.

Table 7-8 and Figure 3-4 show the same property when we change given bargaining power from 0.5 to 1.0.

Table 5: The effect of chinging m
Given $\gamma$=0.5 and $\beta$=0.8.

| m | WF | GDP | |
|---|---|---|---|
| 0.1 | 0.374639 | 0.038051 | PF |
| 0.2 | 0.580907 | 0.045487 | PF |
| 0.3 | 0.65007 | 0.038757 | PF |
| 0.4 | 0.610292 | 0.027046 | PF |
| 0.5 | 0.495969 | 0.015764 | PF |
| 0.6 | | | |
| 0.7 | 0.18056 | 0.002844 | MF |
| 0.8 | 0.080313 | 0.000805 | MF |
| 0.9 | 0.023401 | 0.000132 | MF |

Figure 1: Welfare and GDP

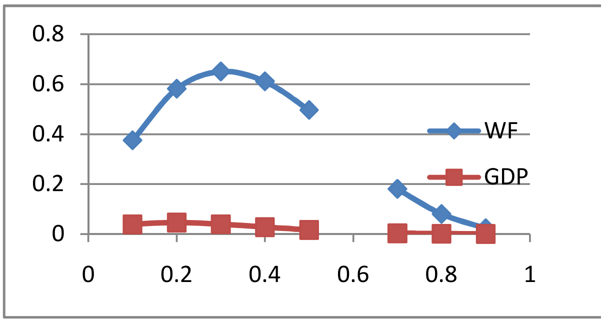



Table 6: The effect of changing $\gamma$
Given m=0.3 and $\beta$=0.8.

| $\gamma$ | WF | GDP | |
|---|---|---|---|
| 0.3 | 0.349453 | 0.010138 | MF |
| 0.37 | 0.349453 | 0.010138 | MF |
| 0.44 | 0.576808 | 0.028077 | PF |
| 0.51 | 0.661775 | 0.040758 | PF |
| 0.58 | 0.738798 | 0.056658 | PF |
| 0.65 | 0.805479 | 0.07603 | PF |
| 0.72 | 0.859558 | 0.098993 | PF |
| 0.79 | 0.899279 | 0.125535 | PF |
| 0.86 | 0.923469 | 0.155465 | PF |
| 0.93 | 0.931704 | 0.188461 | PF |
| 1 | 0.924301 | 0.224073 | PF |

Figure 2: Welfare and GDP

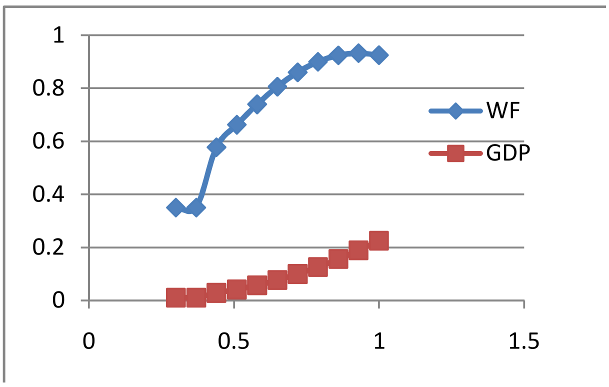


Table 7: : The effect of changing m
Given $\gamma$=1.0 and $\beta$=0.8.

| m | WF | GDP | |
|---|---|---|---|
| 0.1 | 0.393773 | 0.132264 | PF |
| 0.2 | 0.701573 | 0.200826 | PF |
| 0.3 | 0.924301 | 0.224073 | PF |
| 0.4 | 1.05374 | 0.212568 | PF |
| 0.5 | 1.078705 | 0.175795 | PF |
| 0.6 | 0.988176 | 0.124051 | PF |
| 0.7 | 0.777418 | 0.069307 | PF |
| 0.8 | 0.464346 | 0.025041 | PF |
| 0.9 | 0.133755 | 0.002823 | PF |

Figure 3: Welfare and GDP

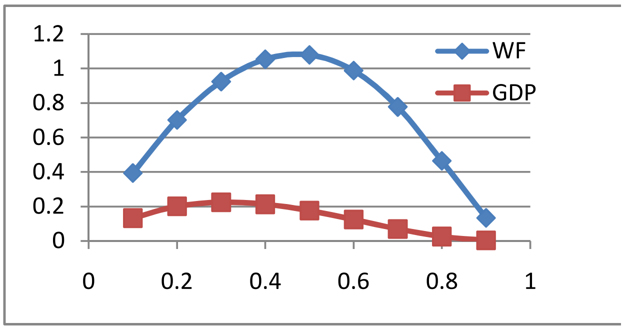
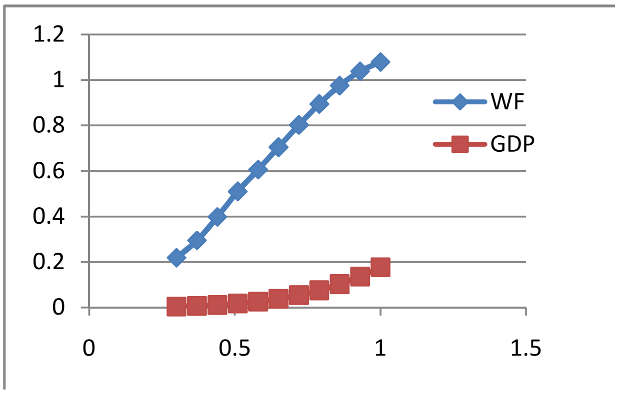



Table 8: The effect of changing $\gamma$
Given m=0.5 and $\beta$=0.8.

| $\gamma$ | WF | GDP | |
|---|---|---|---|
| 0.3 | 0.21786 | 0.003615 | MF |
| 0.37 | 0.293546 | 0.006243 | MF |
| 0.44 | 0.396935 | 0.010445 | MF |
| 0.51 | 0.509298 | 0.016771 | PF |
| 0.58 | 0.605254 | 0.025373 | PF |
| 0.65 | 0.704017 | 0.037289 | PF |
| 0.72 | 0.802002 | 0.053422 | PF |
| 0.79 | 0.894301 | 0.0747 | PF |
| 0.86 | 0.975078 | 0.101949 | PF |
| 0.93 | 1.038258 | 0.135656 | PF |
| 1 | 1.078705 | 0.175795 | PF |

Figure 4: Welfare and GDP




## 5 Conclusion

We analyzes a Trejos-Wright's economy with individual money holdings in the set $\{0, 1, 2\}$, assuming the Nash bargaining rule. We suggest some results as follows: (1) Money supply corresponding to the maximum GDP is smaller than that of the maximum welfare. (2) As the buyer's bargaining power increases, GDP increases. Welfare and the bargaining power have a single-peaked relationship. (3) As the buyer's bargaining power increases, pure-strategy full-support steady states likely exist. (4) There exists the possibility of co-existence of pure-strategy full-support steady states and mixed-strategy full support steady states.

In this paper, we do not examine the stability. It is important to know which steady state equilibria are stable and determinate. Moreover, it is useful to know the effects on the stability by the bargaining power. In addition, The effects on the equilibria by the size of the upper-bound are the issue to be addressed, too.

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