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A Characterization of the Generalized Voting by Issues: Strategy-proof Social Choice Functions under the Single-peaked Preferences with Indifference*

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Abstract

We extend a characterization theorem of the set of social choice functions called *voting by issues* by Nehring and Puppe [12] in two directions: i) We characterize the set of social choice functions called *generalized voting by issues*. ii) The characterization is done without the assumption that prefrences are strict.

Since Nehring and Puppe's framework includes the models in Barberà, Sonnenschein and Zhou [5] (the set of alternatives is the power set of a set), Barberà, Gul and Stacchetti [2] (the set of alternatives is the product of lines) and Danilov [7] (the set of alternatives has a tree structure), our result extends the characterizations in these papers to the case indifference is admitted. While the proof in Nehring and Puppe [12] involves the "reduction" to the subspace, our proof is based on the simple induction argument.

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1 Introduction

The Gibbard-Satterthwaite theorem states that every strategy-proof and onto social choice function is dictatorial when the set of preferences is unrestricted. In contrast to this negative result, it is known that under the restriction of preferences, very rich class of strategy-proof social choice functions emerges. One of the most important restricted preference domains in the literature is the single-peaked domain. The breakthrough of the literature on social choice with single-peaked preferences achieved by Moulin [10]. He identified the set of strategy-proof and onto social choice functions on the line as a generalization of the median voter rule. Since Moulin [10] provides the elegant characterization, the literature on voting uncovered the class of strategy-proof and onto social choice functions under the assumption of single-peaked preferences. For the case the set of alternatives is finite, previous works by Barberà, Sonnenschein and Zhou [5] (the set of alternatives is the set of a power set), Barberà, Gul and Stacchetti [2] (the set of alternatives is the product of lines), Schummer and Vohra [13] (the set of alternatives has a tree structure[1]) and Danilov [7] (the set of alternatives has a tree structure) identify the set of strategy-proof and onto social choice functions under the assumption of single-peakedness. Remarkably, it turned out that strategy-proof and onto social choice functions in these models, especially in first three models, have common structure similar to that had been found by Moulin [10].

Nehring and Puppe [12] unify these models under the abstract framework of *(dichotomic) property space* and *single-peaked preferences*. We follow their wide-range model. Before introducing their framework, let us look at the concept of single-peakedness in the concrete models above. Although each concrete model has a different structure on the set of alternatives, a single-peaked preference in these models is understood as a preference satisfying following two conditions :

i) There exists an ideal alternative p.
ii) If for distinct alternatives x and y, x is at least as "similar" to the ideal alternative p as y is to p, then x is preferred to y.

The "similarity" is the key concept to describe the single-peakedness. It is defined by depending on the structure endowed on the set of alternatives. When the set of alternatives is the line (linearly ordered set), the linear order gives rise to a natural criterion of similarlity: that is, x is at least as similar to p as y is to p if $y \leq x \leq p$ or $p \leq x \leq y$. In the same way, when the set of alternatives is the product of lines, x is at least as similar to p as y is to p if for each coordinate j, $y_j \leq x_j \leq p_j$ or $p_j \leq x_j \leq y_j$. Finally, when the set of alternatives is a tree, x is at least as similar to p as y is to p if x lies on the path from p to y.

Nehring and Puppe provide a general structure on the set of alternatives called a *collection of properties*, and they call the spaces equipped with a collection of properties a *property space*. Since a linearly ordered space, the product of linearly ordered spaces and a tree can be represented by a property space, it is an extreme generalization of the spaces dealt in the literature concerning the single-peaked preferences.

[1] For the precise definition of the tree, see section 4.

In Nehring and Puppe [12], a property is simply represented by a non-empty proper subset of the set of alternatives X. Any elements in the set are considered to have the property, and elements in the complement are not. Roughly speaking, the structure of properties is a collection of non-empty proper subsets of X. Properties are considered to be dichotomic in their model ("left or right", "high or low", and so on.). A property space gives rise to a criterion of the similarlity by the following way: given a collection of properties, x is at least as similar to the ideal alternative p as y is to p *if* y has a property in common with the ideal alternative p, then x also has the property. This expression of the similarlity concept coincides with the natural one discribed above in the concrete models under the apropriate property space representation of the space in question.

On this abstract setting, Nehring and Puppe [12] provide a characterization of the set of social choice functions called *voting by issues*. They show that the set of voting by issues coincides with the set of onto social choice functions satisfying strategy-proofness. Since the models in Barberà, Sonnenschein and Zhou [5], Barberà, Gul and Stacchetti [2] and Danilov [7] are special cases of their general model, the characterization theorem of voting by issues can be seen as a comprehensive result of these concrete models. It is worthwhile to notice that all of these previous results are obtained under the assumption that each voter has a strict preference. But preferences with indifference are quite natural in some concrete models. For example, consider the society facing the problem choosing a location of a public facility (park, library, for instance). Our abstract framework includes two types of this problem. First type is the location problem on the plane, and second type is the location problem on the tree. In these models, for instance, voters may have a preference which only cares about the "distance" from the ideal location. In the plane, the distance from the ideal location to a point is simply measured by the Eucledian metric. In the tree, the distance from the ideal location to a point is measured by the number of vertices on the path between them. While the previous results are obtained by excluding this type of preferences, [2] in this paper, we extend the characterization theorem by Nehring and Puppe [12] to the case that indifference is admitted. Moreover, we characterize the set of social choice functions which we call *generalized voting by issues*. It includes many not necessarily onto social choice functions satisfying strategy-proofness.

A somewhat technical contribution is in proofs. While the proof of the characterization theorem in Nehring and Puppe [12] involves the "reduction" to the subspace, our proof is based on the simple induction argument. Later, we compare the proof technique emplyed in Nehring and Puppe [12] with that of ours.

This paper is organized as follows. In section 2, we introduce the Nehring and Puppe's abstract model. The definition of social choice functions to be characterized and axioms are also given in this section. In section 3, we state our main result. Section 4 offers concrete models captured in our abstract framework. Section 5 concludes the

[2] Schummer and Vohra [13] also consider the location problem on a tree in which this type of preferences are admitted. Since they restrict the attention to the case voters *always* have a preference which cares about the distance from the idel location, their result can not be applied to some important cases. For example, the case in which the set of admissible preferences includes all logically possible single-peaked preferences is excluded in their model. Because of the domain conditions employed in Schummer and Vohra [13] and ours, their results and that of ours are independent.

paper. The appendix offers a proof of Proposition 2.

2 General Model

2.1 Property space and single-peaked preferences

We follow the abstract model proposed in Nehring and Puppe [12]. Let $N = \{1, \cdots, n\}$ be the set of voters. We assume that $2 \leq n < +\infty$. Let X be the set of alternatives. We assume that $2 \leq \#X < +\infty$. A special structure is endowed on X. As mentioned in Introduction, the structure of properties is a collection of non-empty proper subsets of X. In addition to this, we assume that any two alternatives can be distinguished by a property.

Definition. *Let $\mathcal{H} \subseteq 2^X$. The pair $(X, \mathcal{H})$ is a property space if $\mathcal{H}$ satisfies the following conditions H1, H2 and H3.*
H1. $\emptyset \notin \mathcal{H}$.
H2. $\forall H \in \mathcal{H}, H^c \in \mathcal{H}$.
H3. $\forall x, y \in X (x \neq y),\ \exists H \in \mathcal{H} \quad s.t. \quad x \in H \wedge y \in H^c$.

Throughout the paper, except for in the concrete models in section 4, we fix the structure of properties $\mathcal{H}$. The following is the notion of similarity derived from $(X, \mathcal{H})$.

Definition. *For all x, y and z in X, y is at least as **similar** to x as z is to x if*

$$\forall H \in \mathcal{H}, \Big[\{x, z\} \subseteq H \Rightarrow y \in H\Big].$$

The criterion of similarlity induced by $(X, \mathcal{H})$ is the ternary relation defined as follows:

$$T_{\mathcal{H}} = \{(x, y, z) \in X \times X \times X \mid \forall H \in \mathcal{H}, \{x, z\} \subseteq H \Rightarrow y \in H\}$$

For the simplicity, we denote $T_{\mathcal{H}}$ by T. [3] The set of alternatives at least as similar to x as y is to x denoted by $[x, y]$. That is, $[x, y] = \{a \in X \mid (x, a, y) \in T\}$. [4]

[3] The ternary relation T satisfies the following four conditions.
T1 (Reflexivity). $\forall x, y \in X, (x, x, y) \in T \wedge (x, y, y) \in T$.
T2 (Symmetry). $\forall x, y, z \in X, (x, y, z) \in T \Rightarrow (z, y, x) \in T$.
T3 (Transitivity). $\forall x, y, z, x', y' \in X, (x, x', z) \in T \wedge (x, y', z) \in T \wedge (x', y, y') \in T \Rightarrow (x, y, z) \in T$.
T4 (R-antisymmetry). $\forall x, y, z \in X, (x, y, z) \in T \wedge (x, z, y) \in T \Rightarrow y = z$.
In addition to above conditions, by T1 and T4, we have

$$\forall x, y \in X, (x, y, x) \in T \Rightarrow y = x.$$

By T2 and T4, we have L-antisymmetry

$$\forall x, y, z \in X, (x, y, z) \in T \wedge (y, x, z) \in T \Rightarrow x = y.$$

[4] Let $x, y \in X$. The following four statements are true.
(i). $\{x, y\} \subseteq [x, y]$.
(ii). $[x, y] = [y, x]$.
(iii). $\forall u, v \in [x, y], [u, v] \subseteq [x, y]$.
(iv). $x \neq y \Rightarrow \exists z \in [x, y] \backslash \{x\}$ s.t. $[x, z] = \{x, z\}$.

The set of complete and transitive binary relations on X is denoted by $\mathcal{R}$. An element of $\mathcal{R}$ is called a preference. For a preference R, P and I denote the strict part and the indifferent part of R. The set of anti-symmetric preferences is denoted by $\mathcal{P}$. For each $R \in \mathcal{R}$, each non-empty subset $Y \subseteq X$, $\tau(R, Y) = \{x \in Y \mid \forall y \in Y, xRy\}$. That is, $\tau(R, Y)$ is the set of most desirable alternatives in Y with respect to R. For each $R \in \mathcal{R}$, if $\tau(R, X)$ is the singleton, the element of it is denoted by $r_1(R)$. For each $R \in \mathcal{R}$, if R has $r_1(R)$ and $\tau(R, X \backslash \{r_1(R)\})$ is the singleton, the element of it is denoted by $r_2(R)$. The set of admissible preferences is denoted by $\mathcal{D}$. $\mathcal{D}$ is a subset of $\mathcal{R}$.

A **profile of preferences**, or simply a **profile**, is a list of voters' preferences. Profiles are denoted by symbols $\mathbf{R}, \tilde{\mathbf{R}}, \hat{\mathbf{R}}$ etc. The set of admissible profiles is denoted by $\mathcal{D}^N$. For each profile $\mathbf{R} = (R^1, \cdots, R^n)$, each $i \in N$, the subprofile obtained by removing i's preference is denoted by $\mathbf{R}^{-i}$, that is, $\mathbf{R}^{-i} := (R^1, \cdots, R^{i-1}, R^{i+1}, \cdots, R^n)$. It is convenient to write the profile $(R^1, \cdots, R^{i-1}, \hat{R}^i, R^{i+1}, \cdots, R^n)$ as $(\hat{R}^i; \mathbf{R}^{-i})$. A mapping from $\mathcal{D}^N$ to X is called a **social choice function**.

Now, we introduce the notion of single-peakedness. It is characterized by two features; i) it has an ideal alternative, and ii) for all $x, y \in X$, if x is at least as similar to the ideal alternative as y is to the ideal alternative, then x is preferred to y.

Definition. *A preference $R \in \mathcal{R}$ is single-peaked with respect to T if there exists a point, called the peak of R, $p(R) \in X$ such that*

$$\forall x, y \in X (x \neq y), (p(R), x, y) \in T \Rightarrow xPy.$$

The set of single-peaked preferences is denoted by $\mathcal{R}_S$. We assume that admissible preferences are single-peaked: that is, $\mathcal{D} \subseteq \mathcal{R}_S$ [5]. We also assume the following richness conditions R1, R2 and R3 on admissible preference domain.

R1. $\forall x, y \in X (x \neq y),\ [[x, y] = \{x, y\} \Rightarrow \exists R \in \mathcal{D} \text{ s.t. } r_1(R) = x \wedge r_2(R) = y]$.

R2. $\forall x, y, z \in X,\ [z \notin [x, y] \Rightarrow \exists R \in \mathcal{D} \text{ s.t. } r_1(R) = x \wedge yPz]$.

R3. $\forall x \in X,\ \exists R \in \mathcal{D} \cap \mathcal{P} \text{ s.t. } r_1(R) = x$.

R1 and R2 are conditions employed in Nehring and Puppe [12]. They also assume that preferences are strict. Hence, the domain condition employed in Nehring and Puppe [12] is R1, R2 and $\mathcal{D} \subseteq \mathcal{P}$. Obviously, our domain $\mathcal{D}$ satisfying R1, R2 and R3 is not necessarily included in $\mathcal{P}$. Moreover, $\mathcal{D} \cap \mathcal{P}$ does not satisfy the domain condition in Nehring and Puppe [12] in general.

2.2 Voting by issues

A **generalized family of winning coalitions (GFW)** on property $H \in \mathcal{H}$ $\mathcal{W}_H$ is a subset of 2^N satisfying

$$\forall W \in \mathcal{W}_H, \forall W' \subseteq N, \Big[W \subseteq W' \Rightarrow W' \in \mathcal{W}_H\Big].$$

If a **GFW** $\mathcal{W}_H$ satisfies the following condition, we simply call it a **family of winning coalitions (FW)**:

$$\mathcal{W}_H \neq \emptyset \text{ and } \emptyset \notin \mathcal{W}_H.$$

[5] Only in subsection 4.1, we do not assume that admissible preferences are single-peaked preferences.

A **structure of generalized family of winning coalitions (SGFW)** on property space $(X, \mathcal{H})$ is a family of **GFW**s $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ satisfying

$$\forall H \in \mathcal{H}, \forall W \subseteq N, \Big[W \in \mathcal{W}_H \Leftrightarrow W^c \notin \mathcal{W}_{H^c}\Big].$$

If a **SGFW** $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ consists of **FW**s, $\mathcal{W}$ is simply called a **structure of family of winning coalitions (SFW)**.

For each **SGFW** $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$, $f_{\mathcal{W}} : X^n \to 2^X$ is defined as follows: for each $\xi \in X^n$,

$$x \in f_{\mathcal{W}}(\xi) \Leftrightarrow \Big[\forall H \in \mathcal{H}, \Big\{x \in H \Rightarrow \{i \in N \mid \xi_i \in H\} \in \mathcal{W}_H\Big\}\Big].$$

Remark 1. $\forall \xi \in X^n, \# f_{\mathcal{W}}(\xi) \leq 1.$ [6]

Definition. *A **SGFW** $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ is consistent if*

$$\forall \xi \in X^n,\ f_{\mathcal{W}}(\xi) \neq \emptyset.$$

Definition. *Let $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ be a consistent **SGFW**. The generalized voting by issues $F_{\mathcal{W}}$ is defined as follows:*

$$\forall \mathbf{R} \in \mathcal{D}^N, F_{\mathcal{W}}(\mathbf{R}) \in f_{\mathcal{W}}(p(R^1), \cdots, p(R^n)).$$

*If $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ is a consistent **SFW**, $F_{\mathcal{W}}$ is simply called a voting by issues.*

Remark 2. *Let $\mathcal{W} = \{\mathcal{W}_H\}_{H\in\mathcal{H}}$ be a consistent **SGFW**.*

$$\forall H \in \mathcal{H}, \forall \mathbf{R} \in \mathcal{D}^N, \Big[F_{\mathcal{W}}(\mathbf{R}) \in H \Leftrightarrow \{i \in N \mid p(R^i) \in H\} \in \mathcal{W}_H\Big].$$

2.3 Axioms

In this subsection, we introduce our axioms. Let F be our generic notation of social choice function. First axiom requires that one can not be better off by misreporting one's preference.

Strategy-proofness (SP): for all $\mathbf{R} = (R^1, \cdots, R^n) \in \mathcal{D}^N$, all $i \in N$, and all $\tilde{R}^i \in \mathcal{D}$, $F(\mathbf{R})\ R^i F(\hat{R}^i; \mathbf{R}^{-i})$.

Remark 3. *Let a social choice function F be SP. Then, F is unanimous on $F(\mathcal{D}^N)$. That is,*

$$\forall \mathbf{R} \in \mathcal{D}^N,\ \Big[p(R^1) = \cdots = p(R^n) \in F(\mathcal{D}^N) \Rightarrow F(\mathbf{R}) = p(R^1)\Big].$$

Peak-onliness (PO): for all $\mathbf{R}, \tilde{\mathbf{R}} \in \mathcal{D}^N$, $(p(R^1), \cdots, p(R^n)) = (p(\tilde{R}^1), \cdots, p(\tilde{R}^n))$ implies $F(\mathbf{R}) = F(\tilde{\mathbf{R}})$.

Peak-monotonicity (PM): for all $\mathbf{R}, \tilde{\mathbf{R}} \in \mathcal{D}^N$, all $H \in \mathcal{H}$, $F(\mathbf{R}) \in H$ and $\{i \in N \mid p(R^i) \in H\} \subseteq \{i \in N \mid p(\tilde{R}^i) \in H\}$ implies $F(\tilde{\mathbf{R}}) \in H$.

[6] For a proof of remark 1, see Fact 3.1. in Nehring and Puppe [12].

3 Main Result

Proposition 1. *A social choice function F is peak-monotonic if and only if F is a generalized voting by issues.*

Proof. Same as the proof of Proposition 3.1. in Nehring and Puppe [12]. □

Definition. *A subset $Y \subseteq X$ is gated if*

$$\forall x \in X, \exists \gamma_Y(x) \in Y, \forall y \in Y, \gamma_Y(x) \in [x, y].$$

The element $\gamma_Y(x)$ is called the gate of Y to x.[7] The gate $\gamma_Y(x)$ is simply denoted by $\gamma(x)$ when there is no confusion. A gated set Y has an important property such that

$$\forall R \in \mathcal{R}_S,\ \tau(R, Y) = \{\gamma(p(R))\}.$$

An interval $[x, y]$ is an example of a gated set in the models described in subsection 4.2, 4.3 and 4.4. [8] Note that the universal set X is also an example of gated set.

Proposition 2. *Suppose that a social choice function F is strategy-proof. Then, F has gated range if and only if F is peak-only.*

A proof of Proposition 2 is given in Appendix. The following is our characterization of the generalized voting by issues.

Theorem. *A social choice function F which has gated range is strategy-proof if and only if F is a generalized voting by issues.*

Proof. Nehring and Puppe [12] show that *PM* is equivalent to the combination of *SP* and *PO* [9]. Therefore, we have done by Proposition 1 and 2. □

As noted in subsection 2.1, $\mathcal{D}$ does not have subdomain satisfying Nehring and Puppe's domain condition in general. Therefore, the proof of Theorem does not depends on the Nehring and Puppe's characterization theorem. The following is a result also obtained in Nehring and Puppe [12] under the assumption that preferences are strict. As is mentioned in Introduction, since their general framework includes the models in Barberà, Sonnenschein and Zhou [5], Barberà, Gul and Stacchetti [2] and Danilov [7], Corollary 1 extends the characterizations in these papers to the case indifference is admitted. As noted before, since the universal set X is an example of gated set, onto social choice functions satisfy the range conditions in Proposition 2 and Theorem.

Corollary 1. *An onto social choice function F is strategy-proof if and only if F is a voting by issues.*

Note that since our result does **not** exclude the case $\mathcal{D} \subseteq \mathcal{P}$, our result implies Theorem 2 in Nehring and Puppe [12].

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[7] If Y is gated, for all $x, y \in Y$, $[x, y] \subseteq Y$.

[8] The spaces described in these models are examples of *median spaces*. See Nehring and Puppe [12].

[9] See Proposition 3.2. in [12]. Their proof of Proposition 3.2 is valid for the case indifference is admitted.

4 Applications

4.1 The Gibbard-Satterthwaite theorem

In this subsection, we give a generalized result of Corollary 1 in Nehring and Puppe [12]. Throughout this subsection, we assume that $\tilde{\mathcal{D}}$ is a subset of $\mathcal{R}$ that may include non-single-peaked preferences. A social choice function $\tilde{F} : \tilde{\mathcal{D}}^N \to X$ is *strategy-proof if*

$$\forall \mathbf{R} = (R^1, \cdots, R^n) \in \tilde{\mathcal{D}}^N, \forall i \in N, \forall \tilde{R}^i \in \tilde{\mathcal{D}}, \tilde{F}(\mathbf{R})\, R^i \tilde{F}(\hat{R}^i; \mathbf{R}^{-i}).$$

A social choice function $\tilde{F} : \tilde{\mathcal{D}}^N \to X$ is *dictatorial if*

$$\exists i_0 \in N \text{ s.t. } \forall \mathbf{R} = (R^1, \cdots, R^n) \in \tilde{\mathcal{D}}^N, \tilde{F}(\mathbf{R}) \in \tau(R^{i_0}, X).$$

Corollary 2. *Suppose that $\#X \geq 3$ and $(X, \mathcal{H})$ is a property space in which only the universal set and all singletons are gated. Suppose also that $\tilde{\mathcal{D}}$ includes a rich single-peaked preference domain $\mathcal{D}$. If a social choice function $\tilde{F} : \tilde{\mathcal{D}}^N \to X$ is SP and onto, then $\tilde{F}$ is dictatorial.*

Proof. Define $F := \tilde{F}\mid_{\mathcal{D}^N}$. By Corollary 1, F is a voting by issues. Hence, $F = F_{\mathcal{W}}$ for some consistent **SFW** $\mathcal{W}$. We can prove that $F_{\mathcal{W}}$ is dictatorial following the elegant proof for Corollary 1 in Nehring and Puppe [12]. Let $i_0 \in N$ be the dictator of $F_{\mathcal{W}}$.

Next, we prove that i_0 is a dictator of $\tilde{F}$. Suppose the contrary. Assume that there exists $\mathbf{R} = (R^1, \cdots, R^n) \in \tilde{\mathcal{D}}^N$ such that $\tilde{F}(\mathbf{R}) \notin \tau(R^{i_0}, X)$. Let $\tilde{R}^{i_0} \in \mathcal{D}$ with $p(\tilde{R}^{i_0}) \in \tau(R^{i_0}, X)$. If $\tilde{F}(\tilde{R}^{i_0}; \mathbf{R}^{-i_0}) = p(\tilde{R}^{i_0})$, i_0 can manipulate $\tilde{F}$. Hence, $\tilde{F}(\tilde{R}^{i_0}; \mathbf{R}^{-i_0}) \neq p(\tilde{R}^{i_0})$. Let $\tilde{R} \in \mathcal{D}$ be a preference whose peak is $\tilde{F}(\tilde{R}^{i_0}; \mathbf{R}^{-i_0})$. Let $\tilde{\mathbf{R}}^{-i_0}$ be the preference profile other than i_0 in which each voter's preference is $\tilde{R}$. Then, by *SP* of $\tilde{F}$, $\tilde{F}(\tilde{R}^{i_0}; \mathbf{R}^{-i_0}) = \tilde{F}(\tilde{R}^{i_0}; \tilde{\mathbf{R}}^{-i_0}) = F(\tilde{R}^{i_0}; \tilde{\mathbf{R}}^{-i_0})$. This contradicts the fact that i_0 is the dictator of F. □

Barberà and Peleg [4] prove that, in a metric space, if for any $x, y \in X (x \neq y)$, there exists a continuous preference which ranks x first and y second, then a strategy-proof and onto social choice function is dictatorial. Thus, Corollary 2 is an alternative proof of their general result via voting by issues.

4.2 Choosing a subset

Let A be a non-empty finite set. Elements of A are denoted by symbols α, β, γ etc. Barberà, Sonnenschein and Zhou [5] consider the social choice problem in which society chooses a subset of A. The set of alternatives X is the power set of A. That is, $X := 2^A$. Elements of X are denoted by symbols x, y, z etc.

Definition. *For each $\alpha \in A$, $H_\alpha := \{x \in X \mid \alpha \in x\}$. That is, H_α is the collection of subsets of A including α. We define $\mathcal{H} = \{H_\alpha \mid \alpha \in A\} \cup \{H_\alpha^c \mid \alpha \in A\}$.*

It is easy to show that $(X, \mathcal{H})$ is a property space. In this space, for each $p, x, y \in X$, x is at least as similar to p as y is to p *if and only if*

$$\forall \alpha \in A, \Big[\Big\{ \alpha \in p \text{ and } \alpha \in y \Rightarrow \alpha \in x \Big\} \text{ and } \Big\{ \alpha \notin p \text{ and } \alpha \notin y \Rightarrow \alpha \notin x \Big\} \Big].$$

Barberà, Sonnenschein and Zhou [5] show that the set of *SP* and onto social choice functions on $\mathcal{P}^N$ (or the set of strict preferences which have additive representation) is the set of voting by issues[10]. Note that in this model, any **SGFW** are consistent.

4.3 Location problem on the product of lines

4.3.1 Special case: Location problem on the line

Let $X := \{1, 2, \cdots, m\}$. Suppose that m is greater than or equal to 2.

Definition. *For each* $x \in X \backslash \{1\}$, $H^x := \{y \in X \mid y \geq x\}$. *For each* $x \in X \backslash \{m\}$, $H_x := \{y \in X \mid y \leq x\}$. *We define* $\mathcal{H} = \{H^x \mid x \in X \backslash \{1\}\} \cup \{H_x \mid x \in X \backslash \{m\}\}$.

It is easy to verify that $(X, \mathcal{H})$ is a property space. In this space, for each $p, x, y \in X$, x is at least as similar to p as y is to p *if and only if*

$$y \leq x \leq p \;\text{ or }\; p \leq x \leq y.$$

Moulin [10] and Ching [6] show that the set of *SP* and onto social choice functions is the set of voting by issues.

4.3.2 General case

Let m be a natural number. Define $M = \{1, 2, \cdots, m\}$. For each $j \in M$, m_j be a natural number greater than or equal to 2. Let X_j be the set of natural numbers $\{1, 2, \cdots, m_j\}$. Define $X := \Pi_{j \in M} X_j$. For each $j \in M$, let $(X_j, \mathcal{H}_j)$ be the property space defined in the previous subsection. We define $\mathcal{H} = \{H_{j_0} \times \Pi_{j \neq j_0} X_j \mid j_0 \in M, H_{j_0} \in \mathcal{H}_{j_0}\}$.

It is easy to verify that $(X, \mathcal{H})$ is a property space. In this space, for each $p, x, y \in X$, x is at least as similar to p as y is to p *if and only if*

$$\forall j \in M,\; y_j \leq x_j \leq p_j \;\text{ or }\; p_j \leq x_j \leq y_j.$$

Barberà, Gul and Stacchetti [2] show that the set of *SP* social choice functions on $\mathcal{P}^N$ which have gated range is the set of generalized voting by issues. Note that in this model, **SGFW** is consistent *if and only if*

$$\forall H, \tilde{H} \in \mathcal{H}, H \subseteq \tilde{H} \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_{\tilde{H}}.$$

4.4 Location problem on the tree

Let E be a subset of 2^X with $S \in E \Rightarrow \#S = 2$. (X, E) is called a (simple) *graph*. An element of E is called an edge. A sequence $\{x_k\}_{k=1}^K$ in X is a *walk* from x_1 to x_K in (X, E) *if* for each $k \in \{1, 2, \cdots, K-1\}$, $\{x_k, x_{k+1}\} \in E$. A sequence $\{x_k\}_{k=1}^K$ in X is a *path*

[10] Barberà, Sonnenschein and Zhou [5] describe the preferences in $\mathcal{P}$ as separable preferences. In this model, separability and single-peakedness are equivalent when preferences are strict. These two concepts do not coincide in the model with indifference. Ju [9] characterize the set of social choice functions satisfying *SP* and *null-independence* on separable weak preference domain. Although both our model and Ju's model are extension of Barberà, Sonnenschein and Zhou's model to the model with indifference, the domains of preferences are different.

from x_1 to x_K in (X, E) *if* it is a walk satisfying $\#\{x_1, \cdots, x_{k-1}\} = k - 1$. A walk $\{x_k\}_{k=1}^K$ is *closed if* $x_1 = x_K$. A closed path is called a *cycle*.

We assume that (X, E) is a *tree*; that is, following two statements hold.

(i) (*Connectedness*) For all $x, y \in X$, there exists a walk $\{x_k\}_{k=1}^K$ such that $x_1 = x$ and $x_K = y$.

(ii) (*No cycle*) There exists no closed path.

Note that for each $x, y \in X$, there exists an unique path from x to y in this space. Note also that for each edge $e \in E$, the subgraph $(X, E\backslash\{e\})$ has exactly two connected components. That is, for each edge $e = \{x, y\} \in E$, one of the connected components in $(X, E\backslash\{e\})$ includes x and the other component includes y. The former is denoted by H_x^e and the latter is denoted by H_y^e. We define $\mathcal{H} = \{H_x^e \mid e = \{x, y\} \in E\}$.

It is easy to verify that $(X, \mathcal{H})$ is a property space. In this space, for each $p, x, y \in X$, x is at least as similar to p as y is to p *if and only if* x lies on the path from p to y.

Danilov [7] characterizes the set of *SP* and *PO* social choice functions as the set of social choice functions generated by constant mappings and dictatorial mappings.[11] We also characterize the same set by the set of generalized voting by issues without the assumption that admissible preferences are strict. Note that in this model, **SGFW** is consistent *if and only if*

$$\forall H, \tilde{H} \in \mathcal{H}, H \subseteq \tilde{H} \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_{\tilde{H}}.$$

5 Conclusion

In this paper, we extend a characterization theorem of voting by issues by Nehring and Puppe [12] to the case indifference is admitted. Since their framework includes the models in Barberà, Sonnenschein and Zhou [5], Barberà, Gul and Stacchetti [2] and Danilov [7], our result extends the characterizations in these papers to the case indifference is admitted. Moreover, we characterized generalized voting by issues which includes not necessarily onto social choice functions.

Although our theorem is a generalization of many important previous results, it is important to notice that our result does not cover Theorem 3 in Barberà, Massó and Neme [3]. Theorem 3 says that if X is the product of lines and $\mathcal{D} = \mathcal{R}_S \cap \mathcal{P}$, then **any** strategy-proof social choice function can be written as a voting by issues on its range. More precisely, if $F : \mathcal{D}^N \to X$ is strategy-proof, then there exists a consistent **SFW** $\mathcal{W}$ on the subspace $(F(\mathcal{D}^N), \mathcal{H}_{F(\mathcal{D}^N)})$ such that

$$F(\mathbf{R}) = f_{\mathcal{W}}(\tau(R^1, F(\mathcal{D}^N)), \cdots, \tau(R^n, F(\mathcal{D}^N))).^{12}$$

Whether this result holds in Nehring and Puppe's general setting is an open question.

[11] For a precise description, see Danilov [7].

[12] Since each R^i is a strict preference, $\tau(R^i, F(\mathcal{D}^N))$ is a singleton. Here, we abuse the notation to represent the element of $\tau(R^i, F(\mathcal{D}^N))$. Note that $f_{\mathcal{W}}$ is a mapping from $(F(\mathcal{D}^N))^N$ to $F(\mathcal{D}^N)$.

Appendix : Proof of Proposition 2.

In the Appendix, we provide a proof of Proposition 2. The proof follows several lemmas. Lemma 1 says that for any $x, y \in X$, we have a sequence $\{z_k\}_{k=1}^K$ in $[x, y]$ such that the first point is x, and the end point is y, and each point is "adjacent" to the next point; that is, there is no point between them.

Lemma 1. *Let $x, y \in X (x \neq y)$. There exists a finite sequence $\{z_k\}_{k=1}^K$ on $[x, y]$ satisfying the following two conditions.*

(i). $z_1 = x \ \wedge \ z_K = y \ \wedge \ \forall k \in \{1, \cdots, K\} \backslash \{1, K\}, \ z_k \in [z_{k-1}, y] \backslash \{z_{k-1}\}$.

(ii). $\forall k \in \{1, \cdots, K-1\}, [z_k, z_{k+1}] = \{z_k, z_{k+1}\}$.

Proof. Define $z_1 := x$. As noted before,

$$\exists z_2 \in [z_1, y] \backslash \{z_1\} \text{ s.t. } [z_1, z_2] = \{z_1, z_2\}.$$

Suppose that z_k satisfies $z_k \neq y$. Then,

$$\exists z_{k+1} \in [z_k, y] \backslash \{z_k\} \text{ s.t. } [z_k, z_{k+1}] = \{z_k, z_{k+1}\}.$$

For some step K, we have $z_K = y$. Otherwise,

$$[z_1, y] \supsetneq [z_2, y] \supsetneq [z_3, y] \supsetneq \cdots.$$

But this contradicts the fact $[z_1, y]$ is finite. □

For the description of lemmas below, we define the notion of option set originally introduced by Barberà. [13]

Definition. *Let F be a social choice function. For each $i \in N$, and $R^i \in \mathcal{D}$, $\mathbf{R}^{-i} \in \mathcal{D}^{N \backslash \{i\}}$,*

$$O_{-i}^F(R^i) = \{x \in X \mid \exists \tilde{\mathbf{R}}^{-i} \in \mathcal{D}^{N \backslash \{i\}} \text{ s.t. } F(R^i; \tilde{\mathbf{R}}^{-i}) = x\},$$
$$O_i^F(\mathbf{R}^{-i}) = \{x \in X \mid \exists \tilde{R}^i \in \mathcal{D} \text{ s.t. } F(\tilde{R}^i; \mathbf{R}^{-i}) = x\}.$$

Lemma 2 and 3 below are slightly generalized version of Fact B.1. and Lemma B.2. in Nehring and Puppe [12]. Proofs are based on the same technique given by Nehring and Puppe [12].

Lemma 2. *Let $N = \{1, 2\}$. Suppose that a social choice function F is SP and $F(\mathcal{D}^N)$ is gated. Let $i \in N$ and $R^i \in \mathcal{D}$. Let $j \in N \backslash \{i\}$.*

$$\forall x \in O_j^F(R^i), \ \forall y \in [x, p(R^i)] \cap F(\mathcal{D}^N), \ y \in O_j^F(R^i).$$

Proof. Without loss of generality, suppose that $i = 1$ and $j = 2$. Pick $x \in O_2^F(R^1)$ and $y \in [x, p(R^1)] \cap F(\mathcal{D}^N)$ arbitrarily. If $x = y$, we have done. Suppose that $x \neq y$.

Case 1. $[x, y] = \{x, y\}$

[13] For example, see Barberà [1].

By contradiction. Suppose that $y \notin O_2^F(R^1)$. By R1,

$$\exists R_{yx} \in \mathcal{D} \text{ s.t. } r_1(R_{yx}) = y \ \wedge \ r_2(R_{yx}) = x.$$

By *SP* and Remark 3, $F(R^1, R_{yx}) = x$ and $F(R_{yx}, R_{yx}) = y$. Since $y \in [x, p(R^1)] \backslash \{x\}$ and single-peakedness of R^1, yP^1x. Therefore,

$$F(R_{yx}, R_{yx}) P^1 F(R^1, R_{yx}).$$

This contradicts the assumption F is *SP*.

Case 2. $[x, y] \neq \{x, y\}$.

By Lemma 1, we have a sequence $\{z_k\}_{k=1}^K$ on $[x, y]$ ($\subseteq [x, p(R^1)]$) satisfying condition (i) and (ii) in Lemma 1. Since $x, y \in F(\mathcal{D}^N)$ and $F(\mathcal{D}^N)$ is gated, $[x, y] \subseteq F(\mathcal{D}^N)$. Therefore, $\{z_k\}_{k=1}^K$ is a sequence on $[x, p(R^1)] \cap F(\mathcal{D}^N)$. Since $z_1 = x \in O_2^F(R^1)$, by **Case 1** we have $z_2 \in O_2^F(R^1)$. By induction, we have $y = z_K \in O_2^F(R^1)$. □

Lemma 3. *Suppose that a social choice function F is SP and $F(\mathcal{D}^N)$ is gated. Then,*

$$\forall i \in N, \forall R^i, \tilde{R}^i \in \mathcal{D}, \ \left[p(R^i) = p(\tilde{R}^i) \Rightarrow O_{-i}^F(R^i) = O_{-i}^F(\tilde{R}^i) \right].$$

Proof. Without loss of generality, suppose that $i = 1$. Define $x' := p(R^1) (= p(\tilde{R}^1))$ and $x := \gamma(x')$.

Case 1. $N = \{1, 2\}$

By contradiction. Without loss of generality, suppose that

$$\exists y \in O_2^F(R^1) \text{ s.t. } y \notin O_2^F(\tilde{R}^1).$$

Claim 1.1. $x \in O_2^F(\tilde{R}^1)$.

Since $O_2^F(\tilde{R}^1) \neq \emptyset$, we can take an alternative $z \in O_2^F(\tilde{R}^1)$. Because $z \in F(\mathcal{D}^N)$ and $F(\mathcal{D}^N)$ is gated,

$$\gamma(x') \in [z, x'] = [z, p(\tilde{R}^1)].$$

Therefore, we have $x \in [z, p(\tilde{R}^1)] \cap F(\mathcal{D}^N)$. By Lemma 2, $x \in O_2^F(\tilde{R}^1)$. (Proof of Claim 1.1. ends.)

By Claim 1.1., $y \neq x$. By Lemma 1, we have a sequence $\{z_k\}_{k=1}^K$ on $[y, x]$ satisfying the following two conditions.

(i). $z_1 = y \ \wedge \ z_K = x \ \wedge \ \forall k \in \{1, \cdots, K\} \backslash \{1, K\}, \ z_k \in [z_{k-1}, y] \backslash \{z_{k-1}\}$.

(ii). $\forall k \in \{1, \cdots, K-1\}, [z_k, z_{k+1}] = \{z_k, z_{k+1}\}$.

Note that since $y, x \in F(\mathcal{D}^N)$ and $F(\mathcal{D}^N)$ is gated, $z_1, \cdots, z_K \in F(\mathcal{D}^N)$.

Claim 1.2. $z_1, \cdots, z_K \in O_2^F(R^1)$.

Note that $z_1 = y \in O_2^F(R^1)$. Suppose that for some $k (\geq 1)$, $z_k \in O_2^F(R^1)$. Then,

$$z_{k+1} \in [z_k, x] \cap F(\mathcal{D}^N) \subseteq [z_k, p(R^1)] \cap F(\mathcal{D}^N).$$

By Lemma 2, $z_{k+1} \in O_2^F(R^1)$. (Proof of Claim 1.2. ends.)

<u>Claim 1.3</u>. $z_K \notin O_2^F(\tilde{R}^1)$.

Note that $z_1 = y \notin O_2^F(\tilde{R}^1)$. Suppose that for some $k (\geq 1)$, $z_k \notin O_2^F(\tilde{R}^1)$. We prove that $z_{k+1} \notin O_2^F(\tilde{R}^1)$ by contradiction. Suppose the contrary. That is, we assume that $z_{k+1} \in O_2^F(\tilde{R}^1)$. By R1,

$$\exists \tilde{R}^2 \in \mathcal{D} \text{ s.t. } r_1(\tilde{R}^2) = z_k \ \wedge \ r_2(\tilde{R}^2) = z_{k+1}.$$

By *SP* of *F*,

$$F(R^1, \tilde{R}^2) = z_k \ \wedge \ F(\tilde{R}^1, \tilde{R}^2) =_{k+1} .$$

Since $z_{k+1} \in [z_k, p(R^1)] \backslash \{z_k\}$ and single-peakedness of R^1, $z_{k+1} P^1 z_k$. Therefore,

$$F(\tilde{R}^1, \tilde{R}^2) P^1 F(R^1, \tilde{R}^2).$$

This contradicts the assumption *F* is *SP*. (Proof of Claim 1.3. ends.)

Claim 1.3. contradicts the Claim 1.1. (Proof of **Case 1**. ends.)

<u>Case 2</u>. $\# N \geq 3$.

Define $f : \mathcal{D}^2 \to X$ as follows:

$$f(R^1, R^2) = F(R^1, R^2, R^2, \cdots, R^2).$$

Obviously, f is *SP* and $f(\mathcal{D}^2) = F(\mathcal{D}^N)$.

<u>Claim 2.1.</u> $O_{-1}^F(R^1) = O_2^f(R^1)$ and $O_{-1}^F(\tilde{R}^1) = O_2^f(\tilde{R}^1)$.

We prove only the former. Obviously, $O_{-1}^F(R^1) \supseteq O_2^f(R^1)$. Let $z \in O_{-1}^F(R^1)$ be arbitrary. By the definition of $O_{-1}^F(R^1)$, we have a profile $(R^2, \cdots, R^n) \in \mathcal{D}^{N \backslash \{1\}}$ such that $F(R^1, R^2, \cdots, R^n) = z$. Let $R_z \in \mathcal{D}$ be a preference such that $p(R_z) = z$. By *SP* of *F*,

$$\begin{aligned} F(R^1, R^2, \cdots, R^n) &= F(R^1, R_z, R^3, \cdots, R^n) \\ &= F(R^1, R_z, R_z, R^4, \cdots, R^n) \\ &\vdots \\ &= F(R^1, R_z, R_z, \cdots, R_z) \\ &= f(R^1, R_z). \end{aligned}$$

Therefore, $z \in O_2^f(R^1)$. (Proof of Claim 2.1. ends.)

We have

$$\begin{aligned} O_{-1}^F(R^1) &= O_2^f(R^1) && (\because \text{Claim 2.1.}) \\ &= O_2^f(\tilde{R}^1) && (\because \textbf{Case 1}.) \\ &= O_{-1}^F(\tilde{R}^1). && (\because \text{Claim2.1.}) \end{aligned}$$

□

Lemma 4 says that under the assumption of *SP*, *PO* is equivalent to the condition that for any voter i, and any preference profile of other voters $\mathbf{R}^{-i}$, the i's set of options $O_i^F(\mathbf{R}^{-i})$ is gated.

Lemma 4. *Suppose that a social choice function F is SP. F is PO if and only if for all $i \in N$, all $\mathbf{R}^{-i} \in \mathcal{D}^{N\setminus\{i\}}$, $O_i^F(\mathbf{R}^{-i})$ is gated.*

Proof. ($\Rightarrow$) See Lemma3.1. in Nehring and Puppe [12].
($\Leftarrow$) Let $\mathbf{R}, \tilde{\mathbf{R}} \in \mathcal{D}$. Suppose that

$$(p(R^1), \cdots, p(R^n)) = (p(\tilde{R}^1), \cdots, p(\tilde{R}^n)).$$

Without loss of generality, suppose also that $\mathbf{R}^{-1} = \tilde{\mathbf{R}}^{-1}$. Define $x := \gamma(p(R^1))(= \gamma(p(\tilde{R}^1)))$. By the definition of γ, and Lemma 3,

$$\forall y \in O_1^F(R^1)\setminus\{x\}, x \in [y, p(R^1)],$$

$$\forall y \in O_1^F(\tilde{R}^1)\setminus\{x\}, x \in [y, p(\tilde{R}^1)].$$

Since R^1 and $\tilde{R}^1$ are single-peaked preferences,

$$\tau(R^1, O_1^F(R^1)) = \{x\} \ \wedge \ \tau(\tilde{R}^1, O_1^F(\tilde{R}^1)) = \{x\}.$$

Since F is *SP*, $F(R^1; \mathbf{R}^{-1}) = x = F(\tilde{R}^1; \tilde{\mathbf{R}}^{-1})$. □

Lemma 5 says that if a *SP* social choice function has gated range, it satisfies *PO* in two voters case.

Lemma 5. *Let $N = \{1, 2\}$. Suppose that a social choice function F is SP and $F(\mathcal{D}^N)$ is gated. Then, F is PO.*

Proof. Let $i \in N$ and $R^i \in \mathcal{D}$ be arbitrary. We prove that the other voter's option set under R^i is gated. Then, by Lemma 4 we have the conclusion. Without loss of generality, assume that $i = 2$. By R3, we have a preference $\tilde{R}^2 \in \mathcal{D} \cap \mathcal{P}$ such that $p(\tilde{R}^2) = p(R^2)$. By Lemma 3, $O_1^F(\tilde{R}^2) = O_1^F(R^2)$. Hence, we prove that $O_1^F(\tilde{R}^2)$ is gated.

Suppose the contrary. That is,

$$\exists x_0 \in X \ \text{ s.t. } \ \forall x \in O_1^F(\tilde{R}^2), \exists y \in O_1^F(\tilde{R}^2) \ \text{ s.t. } \ x \notin [x_0, y].$$

<u>Claim.</u> $\exists R^1, \tilde{R}^1 \in \mathcal{D}$ s.t. $x_0 = p(R^1) = p(\tilde{R}^1)$ and $F(R^1, \tilde{R}^2) \neq F(\tilde{R}^1, \tilde{R}^2)$.

Let $R^1 \in \mathcal{D}$ be a preference whose peak is x_0. Since $F(R^1, \tilde{R}^2) \in O_1^F(\tilde{R}^2)$, we have $y \in O_1^F(\tilde{R}^2)$ such that

$$F(R^1, \tilde{R}^2) \notin [x_0, y] = [p(R^1), y].$$

By R2, we have a preference $\tilde{R}^1 \in \mathcal{D}$ satisfying $p(\tilde{R}^1) = p(R^1)$ and $y\tilde{P}^1 F(R^1, \tilde{R}^2)$. By *SP*, $F(\tilde{R}^1, \tilde{R}^2) \in \tau(\tilde{R}^1, O_1^F(\tilde{R}^2))$. Hence, $F(R^1, \tilde{R}^2) \neq F(\tilde{R}^1, \tilde{R}^2)$. (Proof of Claim ends.)

Let $z := F(R^1, \tilde{R}^2)$ and $\tilde{z} := F(\tilde{R}^1, \tilde{R}^2)$. Obviously, $z \in O_2^F(R^1)$ and $\tilde{z} \in O_2^F(\tilde{R}^1)$. Therefore, by Lemma 3 $z \in O_2^F(\tilde{R}^1)$ and $\tilde{z} \in O_2^F(R^1)$. By the definition of option sets, we have $\hat{R}^2, \hat{\hat{R}}^2 \in \mathcal{D}$ such that

$$F(\tilde{R}^1, \hat{R}^2) = z \text{ and } F(R^1, \hat{\hat{R}}^2) = \tilde{z}.$$

Since $\tilde{R}^2$ is a strict preference, the following two cases are possible. First, if $z\tilde{P}^2\tilde{z}$, $F(\tilde{R}^1, \hat{R}^2)\tilde{P}^2 F(\tilde{R}^1, \tilde{R}^2)$. This contradicts the *SP* of F. Second, if $\tilde{z}\tilde{P}^2 z$, $F(R^1, \hat{\hat{R}}^2)\tilde{P}^2 F(R^1, \tilde{R}^2)$. This contradicts the *SP* of F. □

Lemma 6 says that if a *SP* social choice function F has gated range, then for any voter i, and a preference R^i, the set of options for other voters is gated. This gated option set result also appears in Nehring and Puppe [12] under the stronger assumption of surjectivity while our assumption is gatedness. This difference is quite important not only for the simple induction proof of Proposition 2, but also for the characterization result for the model with indifference. Later, we compare the proof of Proposition 3.3. in Nehring and Puppe [12] with that of our Proposition 2.

Lemma 6. *Suppose that a social choice function F is SP and $F(\mathcal{D}^N)$ is gated. Then,*

$$\forall i \in N,\ \forall R^i \in \mathcal{D},\ O_{-i}^F(R^i) \text{ is gated.}$$

Proof. Let $i \in N, R^i \in \mathcal{D}$. Without loss of generality, suppose that $i = 1$. Define $f : \mathcal{D}^2 \to X$ as follows:

$$f(R^1, R^2) = F(R^1, R^2, R^2, \cdots, R^2).$$

Obviously, f is *SP* and $f(\mathcal{D}^2)(= F(\mathcal{D}^N))$ is gated. It is easy to see that $O_{-1}^F(R^1) = O_2^f(R^1)$. By Lemma 5, f is *PO*. By Lemma 4, $O_2^f(R^1)$ is gated. Therefore, $O_{-1}^F(R^1)$ is gated. □

Now, we provide a proof of Proposition 2. The proof of Proposition 3.3. in Nehring and Puppe [12], which states that any *SP* and onto social choice function satisfy *PO* , is done by the induction for the number of voters;

First, they prove two voters case. To prove the general case, let $R^1, \tilde{R}^1 \in \mathcal{D}$ satisfy $p(R^1) = p(\tilde{R}^1)$. Define $F_1, \tilde{F}_1$ by F fixing voter 1's preference as R^1 and $\tilde{R}^1$, respectively. Then, they prove the general case by showing $F_1 = \tilde{F}_1$. Because F_1 and $\tilde{F}_1$ may not surjective, to apply induction hypothesis they restrict the domain of F_1 and $\tilde{F}_1$ to the preferences on $F_1(\mathcal{D}^{N\setminus\{1\}})$ $(= \tilde{F}_1(\mathcal{D}^{N\setminus\{1\}}))$. That is, define $\mathcal{D}_1 := \{R\ |_{F_1(\mathcal{D}^{N\setminus\{1\}})} |\ R \in \mathcal{D}\}$. Then, define $F_1' : \mathcal{D}_1^{N\setminus\{1\}} \to F_1(\mathcal{D}^{N\setminus\{1\}})$ and $\tilde{F}_1' : \mathcal{D}_1^{N\setminus\{1\}} \to F_1(\mathcal{D}^{N\setminus\{1\}})$ by

$$F_1'(R^2\ |_{F_1(\mathcal{D}^{N\setminus\{1\}})}, \cdots, R^n\ |_{F_1(\mathcal{D}^{N\setminus\{1\}})}) = F_1(R^2, \cdots, R^n),$$

$$\tilde{F}_1'(R^2\ |_{F_1(\mathcal{D}^{N\setminus\{1\}})}, \cdots, R^n\ |_{F_1(\mathcal{D}^{N\setminus\{1\}})}) = \tilde{F}_1(R^2, \cdots, R^n).$$

In this operation, their assumption that admissible preferences are strict is needed to guarantee the well-definedness of F_1' and $\tilde{F}_1'$. In our proof of Proposition 2, since we relax the surjectivity, the reduction argument to the range of F_1 above is avoided.

Proof of Proposition 2. ($\Rightarrow$) Let $i \in N, \mathbf{R}^{*-i} \in \mathcal{D}^{N\setminus\{i\}}$. Without loss of generality, suppose that $i = n$. Define $F_0 := F$. For each $k = 1, \cdots, n-2$, define $F_k : \mathcal{D}^{N\setminus\{1,\cdots,k\}} \to X$ as follows:

$$\forall \mathbf{R}^{-\{1,\cdots,k\}} \in \mathcal{D}^{N\setminus\{1,\cdots,k\}},\ F_k(\mathbf{R}^{-\{1,\cdots,k\}}) = F_{k-1}(R^{*k}; \mathbf{R}^{-\{1,\cdots,k\}}).$$

Claim. $F_0,\ F_1,\ \cdots,\ F_{n-2}$ are *SP* social choice functions whose ranges are gated.
Proof of Claim. F_0 is *SP* and $F_0(\mathcal{D}^N)$ is gated. Suppose that F_k is *SP* and $F_k(\mathcal{D}^{N\setminus\{1,\cdots,k\}})$ is gated. Obviously, F_{k+1} is *SP*. By the definition of F_{k+1},

$$F_{k+1}(\mathcal{D}^{N\setminus\{1,\cdots,k+1\}}) = O^{F_k}_{-(k+1)}(R^{*k+1}).$$

By Lemma 6, the right side of the equation is gated. (Proof of Claim ends.)

By the Claim above, F_{n-2} is *SP* and $F_{n-2}(\mathcal{D}^{\{1,\cdots,n-2\}})$ is gated. By Lemma 5, F_{n-2} is *PO*. Thus, $O^{F_{n-2}}_n(R^{*n-1})$ is gated. By the definition of $F_0,\ F_1,\ \cdots,\ F_{n-2}$,

$$\begin{aligned} O^{F_{n-2}}_n(R^{*n-1}) &= O^{F_{n-3}}_n(R^{*n-2}, R^{*n-1}) \\ &= O^{F_{n-4}}_n(R^{*n-3}, R^{*n-2}, R^{*n-1}) \\ &\vdots \\ &= O^{F_1}_n(R^{*2}, R^{*3}, \cdots, R^{*n-1}) \\ &= O^{F_0}_n(R^{*1}, R^{*2}, \cdots, R^{*n-1}) \\ &= O^{F}_n(\mathbf{R}^{*-n}) \end{aligned}$$

Therefore, $O^F_n(\mathbf{R}^{-n})$ is gated. By Lemma 4, F is *PO*.
($\Leftarrow$) By Proposition 1, F is *PM*. Let $x \in X$. Let $R_x \in \mathcal{D}$ be a preference satisfying $p(R_x) = x$. Define $\mathbf{R} := (R_x, \cdots, R_x)$ and

$$\gamma(x) = F(\mathbf{R}).$$

Let $y \in F(\mathcal{D}^N)$ be arbitrary. We prove $\gamma(x) \in [x, y]$ by contradiction. Suppose that $(x, \gamma(x), y) \notin T$. Then, by the definition of T,

$$\exists H \in \mathcal{H} \text{ s.t. } \{x, y\} \subseteq H \ \wedge\ \gamma(x) \notin H.$$

Note that $F(\mathbf{R}) = \gamma(x) \notin H$. Let $R_y \in \mathcal{D}$ be a preference satisfying $p(R_y) = y$. Define $\tilde{\mathbf{R}} := (R_y, \cdots, R_y)$. By Lemma 3, $F(\tilde{\mathbf{R}}) = y \in H$. Obviously, $N = \{i \in N \mid p(\tilde{R}^i) \in H\} = \{i \in N \mid p(R^i) \in H\}$. Therefore, by *PM* of F, $F(\mathbf{R}) \in H$, a contradiction. □

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