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A time allocation analysis of road transport in the monocentric cities

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Abstract

We consider the queuing pattern at some possible bottleneck locations in cities and applicable conditions of time-variant congestion charge for continuous land use. The ADL model shows road user's departure time choice and charging rule which re-allocate each road user's departure time. And, this model has been extended for discrete networks. However, the settings of these analysis included some contradictions. In addition, discrete settings have a difficulty which is to extend into land use optimization. To resolve these problems, we reconstruct two-tandem bottleneck model and show the equilibrium and the welfare changes. Besides, we introduce continuous land into the ADL model. We show that applicable conditions depend on maximum traffic volume/capacity and capacity of other point.

Key Words: ADL model, two-tandem bottlenecks, "Paradox", monocentric city, continuous land, traffic volume/capacity and capacity.

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1 Introduction

We consider the queuing pattern at some possible bottleneck locations in cities and the applicable conditions of time-variant congestion charge for continuous land use. Arnott, de Palma and Lindsey (1988, 1990a) shows the road user's departure time choice and charging rule which re-allocate each road user's departure time to maximize the road user's surplus (hereafter ADL model). The ADL model depicts the situation that individuals use a capacity-fixed bottleneck and traffic over-flow in peak period. However, the original ADL model does not consider spacial features. Because the objective of transport is to travel, it is necessary to consider the relationship between congestion and spacial features. Practically, where to charge is a matter of great importance in a congested city.

Several studies are attempted to introduce spacial feature in the ADL model using networks. ADL (1990b, 1993b) consider route choices and optimal tolls in two route networks. Kuwahara (1990) solves two-tandem bottleneck problem. However, Kuwahara (1990) assumes distribution of desire departure time as given. A distribution of desire departure time must be induced by optimization of each commuter. ADL (1990b) consider route choices and optimal tolls in two routes, and ADL (1993b) consider queuing patterns and optimal tolls in "Y-shape" network. However, the setting of ADL (1993b) is substantially same with Kuwahara (1990). ADL (1993b) analyze a welfare between numbers of commuters and capacity sizes. And ADL (1993b) also examine the change of total surplus with upstream bottleneck capacity expansion. They concluded that upstream bottleneck capacity expansion may lead to a decrease on total surplus. ADL (1993b) named this phenomenon as "A dynamic traffic equilibrium paradox". However, The sorting criterion in ADL (1993b) is inappropriate. The criterion in ADL (1993b) has a contradiction among equilibrium costs of two commuter groups, bottleneck capacities and numbers of commuters. For example, "equilibrium cost of upstream commuters" > "that of downstream commuters" contradicts "upstream capacity" > "downstream capacity". Daniel, Gisches and Rapoport (2009) points out the "Paradox" again and constructs a experiment model However, Daniel, Gisches and Rapoport (2009) are based on ADL (1993b). Therefore, we re-formulate two-tandem bottleneck problem and show that the "Paradox" never occur.

On the other hand, in an urban district, due to a scarcity of land, we should consider the efficient allocation of land. However, in the literature of urban economics, it is general to assume continuity on land. Therefore, to consider the relationship between congestion and land use, we will show applicable conditions and charging rule based on the ADL model for continuous land. However, in this paper, we assume the distribution of residents and land use patterns are fixed.

2 Model

We begin with characterizing spacial features. We consider the monocentric city which has congested roads. The center of the city is r_c, and the boundary is $f_f.(r_c < r_f)$. The city has fixed population $N \in \mathbb{R}$. Suppose that the people who live in this city are homogeneous, and commute for the CBD (Central Business District). In the CBD, there is no congestion. Land is allocated for housing and transport, and the allocation between housing and transport is fixed. Each distance $r \in \mathbb{R}$ (from the CBD), $n(r)$ is located($n(r) > 0$). Therefore, we can get

$$N(r) = \int_r^{r_f} n(x)dx. \tag{1}$$

$N(r)$ monotonically increases when r decreases. The total amount of land and the land for transport at any point are denoted $L(r)$, $L_T(r)$ ($L_T(r) \in \mathbb{R}$, $L_T(r)$ is continuous on r and $0 < L_T(r) < L(r)$). Also we assume that the capacity of road is equivalent to $L_T(r)$.

In this paper, to search the charging points, we adopt the ratio of traffic volume to capacity of road ($\frac{N(r)}{L_T(r)}$) as an indicator of the degree of congestion. At the locations where the ratio of traffic volume to capacity of road is over 1 ($\frac{N(r)}{L_T(r)} > 1$), congestions may occur because the traffic volumes over the road capacities. Therefore, these locations are possible bottleneck locations. For $\frac{N(r)}{L_T(r)}$, since $[r_c, r_f]$ is compact and continuous, there exist maximum of $\frac{N(r)}{L_T(r)}$ on $[r_c, r_f]$ (if some points take same value of $\frac{N(r)}{L_T(r)}$, we take the nearest one to the CBD). Then, we will call this r as $r_m^1 \in [r_c, r_f]$. Furthermore, for $[r_c, r_m^1)$, it does not satisfy compactness, we can't say the existence of maximum. However if there exist, we call this r as $r_m^2 \in [r_c, r_m^1)$ and so (the $i-th$ largest $\frac{N(r)}{L_T(r)}$ is named r_m^i).

Next, we characterize chronic features, and these are almost the same as in the original ADL model. We pick any r_m^i from $[r_c, r_f]$, and consider the commuter trips $N(r_m^i)$ from each locations far from r_m^i to the CBD (commuters have the same fixed arrival time t^*). If traffic volumes reach the road capacity at r_m^i(which is $L_T(r_m^i)$), a queue is formed. Each commuter must choose how long he will waste time in the queue or in the CBD. The cost of waiting time in the CBD is called schedule delay cost. On the other hand, the cost he must bear in queue is called travel time cost. Hence, we can present trip cost as

$$C(t) = \alpha(travel\ time) + \beta(time\ early) + \gamma(time\ late), \tag{2}$$

where α, β, γ are each shadow cost and t is departure time for each commuter. For simplicity, we assume that travel time $T(r_m^i)$ consist of waiting time in a queue (i.e., travel time by free-flow is 0). Therefore, we can denote each trip cost as $t^* - t - T(t)$ or $t + T(t) - t^*$.

The no-toll equilibrium is the *Nash equilibrium* which is attained by each commuter who decides his departure time under the decisions of others. For each time, the bottleneck can handle $L_T(r)$ traffic volume, hence, we can get $t_e - t_0 = N(r)/L_T(r)$. If the commuters depart at t_0, they will pay the trip price $p(t_0) = C(t_0) = \beta(t^* - t_0)$ and if the commuters depart at t_e, they will pay the trip price $p(t_e) = C(t_e) = \gamma(t_e - t^*)$. In equilibrium, all commuters must have equal trip price, hence, from $p(t_0) = p(t_n) = p(t_e)$, we can get

$$t_0 = t^* - \frac{\gamma}{\beta+\gamma}\frac{N(r)}{L_T(r)}, t_n = t^* - \frac{\beta\gamma}{\alpha(\beta+\gamma)}\frac{N(r)}{L_T(r)}, t_e = t^* + \frac{\beta}{\beta+\gamma}\frac{N(r)}{L_T(r)}. \tag{3}$$

And we have trip price

$$p = C = t^* - \frac{\beta\gamma}{\beta+\gamma}\frac{N(r)}{L_T(r)}. \tag{4}$$

Now, we can lead the toll for clearing the queue and maximizing the social surplus :

$$\tau(t,r) = \begin{cases} \frac{\beta\gamma}{\beta+\gamma}\frac{N(r)}{L_T(r)} - \beta(t^* - t) & if \quad t \in [t_0, t^*] \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N(r)}{L_T(r)} - \gamma(t - t^*) & if \quad t \in [t^*, t_e] \end{cases} . \tag{5}$$

This toll is corresponding to the waiting cost in the queue at each time. By this toll, we can maximize the social surplus without changing the interval of peak period. Therefore, we can recover the queuing time in monetary term which is dead weight loss. Next, we will consider the no-toll equilibrium in multiple bottlenecks under this framework.

3 The no-toll equilibrium in multiple bottlenecks

At first, for simplification, we examine the two bottleneck cases in discrete locations. We assume the two bottlenecks L_T^A, L_T^B (L_T^A is near the CBD) and the commuter groups n_A, n_B which locate L_T^A, L_T^B each other ($n_A + n_B = N$). And we assume that each bottleneck satisfies $\frac{N}{L_T^A} > 1$, $\frac{N}{L_T^B} > 1$. The number of departing commuters at each time $t \in [t_q^i,\, t_{q'}^i]$ are defined $x_i(t)$ $(i = A, B)$. Then, the queue length at L_T^B is

$$D_x^B(t) = \begin{cases} \int_{t_q^B}^t x_B(u)du - L_T^B(t - t_q^B) & if \quad \int_{t_q^B}^t x_B(u)du > L_T^B(t - t_q^B) \\ 0 & if \quad otherwise \end{cases} . \tag{6}$$

Therefore, the commuters who start at time t encounter the waiting time $T_x^B(t) = \frac{D_x^B(t)}{L_T^B}$. On the other hand, the queue length at L_T^A is

$$D_x^A(t) = \begin{cases} \int_{t_q^A}^{\hat{t}} x_A(u)du + L_T^B(\hat{t} - t_q^A) - L_T^A(\hat{t} - t_q^A) \\ \qquad if \quad \int_{t_q^A}^{\hat{t}} x_A(u)du + L_T^B(\hat{t} - t_q^A) > L_T^A(\hat{t} - t_q^A) \\ 0 \qquad if \qquad otherwise \end{cases} , \tag{7}$$

but $\hat{t} = t + T_x^B(t)$. Therefore, the commuters who start at time t, located at L_T^B and who start at time $\hat{t}$, located at L_T^A encounter the waiting time $T_x^A(t) = \frac{D_x^A(t)}{L_T^A}$.

Next, we summarize four cases of queuing pattern of two bottlenecks;

$$\begin{cases} D_x^A(t) = 0 & \wedge & D_x^B(t) = 0 & \cdots\cdots(i) \\ D_x^A(t) > 0 & \wedge & D_x^B(t) = 0 & \cdots\cdots(ii) \\ D_x^A(t) = 0 & \wedge & D_x^B(t) > 0 & \cdots\cdots(iii) \\ D_x^A(t) > 0 & \wedge & D_x^B(t) > 0 & \cdots\cdots(iv) \end{cases} , \tag{8}$$

$$\Rightarrow \quad \begin{array}{l} (i)\left\{\begin{array}{l} \dot{D}_x^A(t)=0 \\ \dot{D}_x^B(t)=0 \end{array}\right. \\ \qquad \Rightarrow \left\{\begin{array}{l} \dot{T}_x^A(t)=\dot{T}_x^A(\hat{t})=0 \\ \dot{T}_x^B(t)=0 \end{array}\right. \\ (ii)\left\{\begin{array}{l} \dot{D}_x^A(t)=x_A(t)+x_B(t)-L_T^A \\ \dot{D}_x^B(t)=0 \end{array}\right. \\ \qquad \Rightarrow \left\{\begin{array}{l} \dot{T}_x^A(t)=\dot{T}_x^A(\hat{t})=\frac{x_A(t)+x_B(t)-L_T^A}{L_T^A} \\ \dot{T}_x^B(t)=0 \end{array}\right. \\ (iii)\left\{\begin{array}{l} \dot{D}_x^A(t)=0 \\ \dot{D}_x^B(t)=x_B(t)-L_T^B \end{array}\right. \\ \qquad \Rightarrow \left\{\begin{array}{l} \dot{T}_x^A(t)=\dot{T}_x^A(\hat{t})=0 \\ \dot{T}_x^B(t)=\frac{x_B(t)-L_T^B}{L_T^B} \end{array}\right. \\ (iv)\left\{\begin{array}{l} \dot{D}_x^A(\hat{t})=x_A(\hat{t})+L_T^B-L_T^A \\ \dot{D}_x^A(t)=\left(1+\dot{T}_x^B(t)\right)\left(x_A(\hat{t})+L_T^B-L_T^A\right) \\ \dot{D}_x^B(t)=x_B(t)-L_T^B \end{array}\right. \\ \qquad \Rightarrow \left\{\begin{array}{l} \dot{T}_x^A(\hat{t})=\frac{x_A(\hat{t})+L_T^B-L_T^A}{L_T^A} \\ \dot{T}_x^A(t)=\frac{\left(1+\dot{T}_x^B(t)\right)\left(x_A(\hat{t})+L_T^B-L_T^A\right)}{L_T^B} \\ \dot{T}_x^B(t)=\frac{x_B(t)-L_T^B}{L_T^B} \end{array}\right. \end{array} . \tag{9}$$

The differences between $\dot{T}_x^A(t)$ and $\dot{T}_x^A(\hat{t})$ in the case (iv) derives from the timing differences of joining the queue $D_x^A(t)$ of commuters locating at L_T^A and L_T^B. Commuters who locate at L_T^A (i.e., n_A) join the $D_x^A(t)$ at the same time as they depart, on the other hand, who locate at L_T^B (i.e., n_B) join the $D_x^A(t)$ after passing through L_T^B. Hence, commuters of n_A face the queue length $D_x^A(\hat{t})$ and commuters of n_B face the queue length $D_x^A(t)$. These two equations will be required to find the value of $x_i(t)$ which satisfy the first order condition of n_B.

Next, we set the commuters' problem. The n_A's problem is cost minimization given α,β,$\gamma > 0$, t^*, $T_x^A(t) = \frac{D_x^A(t)}{L_T^A}$, $x_A(t) > 0$, i.e.,

$$\min_t C_A(t) = \left\{\begin{array}{lll} \alpha T_x^A(t) + \beta(t^* - t - T_x^A(t)) & if & t \leq \widetilde{t}_A \\ \alpha T_x^A(t) + \gamma(t + T_x^A(t) - t^*) & if & t > \widetilde{t}_A \end{array}\right. , \tag{10}$$

If $D_x^B(t) > 0$, $T_x^A(\cdot)$ is not t then $\hat{t}$. Similarly, the n_B's problem is also cost minimization given α, β, $\gamma > 0$, t^*, $T_x^B(t) = \frac{D_x^B(t)}{L_T^B}$, $x_B(t) > 0$, i.e.,

$$\min_t C_B(t) = \left\{\begin{array}{lll} \alpha(T_x^A(t) + T_x^B(t)) + \beta(t^* - t - T_x^A(t) - T_x^B(t)) & if & t \leq \widetilde{t}_B \\ \alpha(T_x^A(t) + T_x^B(t)) + \gamma(t + T_x^A(t) + T_x^B(t) - t^*) & if & t > \widetilde{t}_B \end{array}\right. . \tag{11}$$

From these problems, we can get next relations (first order conditions : hereafter FOC),

$$\dot{T}_x^A(t) = \begin{cases} \frac{\beta}{\alpha-\beta} & if \quad t \leq \widetilde{t}_A \\ -\frac{\gamma}{\alpha+\gamma} & if \quad t > \widetilde{t}_A \end{cases} \tag{12}$$

and

$$\dot{T}_x^B(t) = \begin{cases} \frac{\frac{\beta}{\alpha-\beta} - \dot{T}_x^A(t)}{1+\dot{T}_x^A(t)} & if \quad t \leq \widetilde{t}_B \\ -\frac{\frac{\gamma}{\alpha+\gamma} + \dot{T}_x^A(t)}{1+\dot{T}_x^A(t)} & if \quad t > \widetilde{t}_B \end{cases} . \tag{13}$$

Then, we will find the value of $x_i(t)$ which satisfy the FOC (case (i) is trivial, hence we omit this case).

$$(ii) \quad x_A(t) = \begin{cases} \frac{\beta}{\alpha-\beta} L_T^A - x_B(t) + L_T^A & if \quad t \leq \widetilde{t}_A \\ -\frac{\gamma}{\alpha+\gamma} L_T^A - x_B(t) + L_T^A & if \quad t > \widetilde{t}_A \end{cases} , \tag{14a}$$

$$x_B(t) = \begin{cases} \frac{\beta}{\alpha-\beta} L_T^A - x_A(t) + L_T^A & if \quad t \leq \widetilde{t}_B \\ -\frac{\gamma}{\alpha+\gamma} L_T^A - x_A(t) + L_T^A & if \quad t > \widetilde{t}_B \end{cases} . \tag{14b}$$

$$(iii) \quad \nexists x_A(t) \ which \ satisfy \ the \ FOC, \tag{15a}$$

$$x_B(t) = \begin{cases} \frac{\beta}{\alpha-\beta} L_T^B + L_T^B & if \quad t \leq \widetilde{t}_B \\ -\frac{\gamma}{\alpha+\gamma} L_T^B + L_T^B & if \quad t > \widetilde{t}_B \end{cases} . \tag{15b}$$

$$(iv) \quad x_A(\hat{t}) = \begin{cases} \frac{\beta}{\alpha-\beta} L_T^A + L_T^A - L_T^B & if \quad \hat{t} \leq \widetilde{t}_A \\ -\frac{\gamma}{\alpha+\gamma} L_T^A + L_T^A - L_T^B & if \quad \hat{t} > \widetilde{t}_A \end{cases} , \tag{16a}$$

$$x_B(t) = L_T^B \qquad for \ all \ t. \tag{16b}$$

In these cases, from assumptions and FOC, the possible patterns are only pattern (ii) and pattern $(iii)-(iv)-(iii)$ (proof in appendix 1). Pattern (ii) is equivalent to original ADL model at L_T^A for N persons. Hence, we will show the equilibrium of pattern $(iii)-(iv)-(iii)$. The equilibrium of pattern $(iii)-(iv)-(iii)$ is constituted by each (L_T^A and L_T^B) queuing time interval and switching times from (iii) to (iv) and (iv) to (iii). To induce the equilibrium, we will show the conditions,

$$\bar{t} = t_q^A - T_x^B(\bar{t}), \tag{17a}$$

$$\bar{\bar{t}} = t_{q'}^A - T_x^B(\bar{\bar{t}}), \tag{17b}$$

$$\tilde{t}_A = t^* - T_x^A(\tilde{t}_A), \tag{17c}$$

$$\tilde{t}_B = t^* - T_x^A(\tilde{t}_A) - T_x^B(\tilde{t}_B) = \tilde{t}_A - T_x^B(\tilde{t}_B), \tag{17d}$$

$$\int_{t_q^A}^{t_{q'}^A} x_A(u)du = \left(\tilde{t}_A - t_q^A\right)\left(\frac{\beta}{\alpha-\beta}L_T^A - L_T^B + L_T^A\right) + \left(t_{q'}^A - \tilde{t}_A\right)\left(-\frac{\gamma}{\alpha+\gamma}L_T^A - L_T^B + L_T^A\right) = n_A, \tag{17e}$$

$$\int_{t_q^B}^{t_{q'}^B} x_B(u)du = \left(\bar{t} - t_q^B\right)\left(\frac{\beta}{\alpha-\beta}L_T^B + L_T^B\right) + \left(\bar{\bar{t}} - \bar{t}\right)L_T^B + \left(t_{q'}^B - \bar{\bar{t}}\right)\left(-\frac{\gamma}{\alpha+\gamma}L_T^B + L_T^B\right) = n_B, \tag{17f}$$

$$\left(\tilde{t}_A - t_q^A\right)\frac{\beta}{\alpha-\beta}L_T^A = \left(t_{q'}^A - \tilde{t}_A\right)\frac{\gamma}{\alpha+\gamma}L_T^A \qquad at \quad L_T^A, \tag{17g}$$

$$\left(\bar{t} - t_q^B\right)\frac{\beta}{\alpha-\beta}L_T^B = \left(t_{q'}^B - \bar{\bar{t}}\right)\frac{\gamma}{\alpha+\gamma}L_T^B \qquad at \quad L_T^B, \tag{17h}$$

$$t_q^A - \bar{t} = \tilde{t}_A - \tilde{t}_B = t_{q'}^A - \bar{\bar{t}}. \tag{17i}$$

where, each $\bar{t}$ and $\bar{\bar{t}}$ denotes the departure time of n_B when n_B can reach L_T^A at t_q^A and $t_{q'}^A$. From these conditions, we can get next solutions[1],

[1] We can prove that these solutions lead the equilibrium of n_B from equality between $C_B(t) = \frac{\beta\gamma}{\beta+\gamma}\frac{n_B}{L_T^B}$ $if\ t \in [t_q^B, \bar{t}]$ and $C_B(t) = \frac{\beta\gamma}{\beta+\gamma}\frac{n_A}{L_T^A - L_T^B} - \alpha(t_q^A - \bar{t})\ if\ t \in [\bar{t}, \bar{\bar{t}}]$.

$$t_q^A = t^* - \frac{n_A}{L_T^A - L_T^B}\frac{\gamma}{\beta+\gamma}, \tag{18}$$

$$t_{q'}^A = t^* + \frac{n_A}{L_T^A - L_T^B}\frac{\beta}{\beta+\gamma}, \tag{19}$$

$$\tilde{t}_A = t^* - \frac{n_A}{L_T^A - L_T^B}\frac{\beta\gamma}{\alpha(\beta+\gamma)}, \tag{20}$$

$$t_q^B = t^* - \frac{n_B}{L_T^B}\frac{\gamma}{\beta+\gamma}, \tag{21}$$

$$t_{q'}^B = t^* + \frac{n_B}{L_T^B}\frac{\beta}{\beta+\gamma}, \tag{22}$$

$$\tilde{t}_B = t^* - \frac{n_B}{L_T^B}\frac{\beta\gamma}{\alpha(\beta+\gamma)}, \tag{23}$$

$$\bar{t} = t^* - \frac{n_A}{L_T^A - L_T^B}\frac{\gamma(\alpha-\beta)}{\alpha(\beta+\gamma)} - \frac{n_B}{L_T^B}\frac{\beta\gamma}{\alpha(\beta+\gamma)}, \tag{24}$$

$$\bar{\bar{t}} = t^* + \frac{n_A}{L_T^A - L_T^B}\frac{\beta(\alpha+\gamma)}{\alpha(\beta+\gamma)} - \frac{n_B}{L_T^B}\frac{\beta\gamma}{\alpha(\beta+\gamma)}. \tag{25}$$

This result implies that pattern $(iii) - (iv) - (iii) \Leftrightarrow \frac{n_A}{L_T^A - L_T^B} < \frac{n_B}{L_T^B} \Leftrightarrow \frac{N}{L_T^A} < \frac{n_B}{L_T^B}$ otherwise pattern (ii) will realize. Namely, if the possible bottleneck location near the CBD has lower traffic volume over capacity (hereafter traffic volume/capacity) than far the CBD, both possible bottleneck locations become bottlenecks. On the other hand, if the possible bottleneck location near the CBD has high traffic volume/capacity than far the CBD, only this possible bottleneck location become a bottleneck. Therefore, to search the bottlenecks (i.e., charging points) according to the value of traffic volume/capacity (from large to small) ensures no existence of bottleneck far from such a point (or in the interval from i-th largest bottleneck to $i+1$-th largest bottleneck).

Proposition 1 *In the monocentric city, we suppose that there are two possible bottleneck locations in which the traffic volume excess the capacity. Then, whether both two possible bottleneck locations become bottlenecks or only one possible bottleneck location becomes bottleneck depend on the relative sizes of traffic volume/capacity at two locations. If the possible bottleneck location nearer the CBD has less traffic volume/capacity, then both possible bottleneck locations become bottlenecks. On the other hand, if the possible bottleneck location nearer the CBD has much traffic volume/capacity, only this possible bottleneck location becomes a bottleneck.*

3.1 Equilibrium user's costs and capacity expansion

We examine the equilibrium commuters' travel costs, total cost of each commuter group and the changes of total costs according to capacity expansion in upstream bottleneck in pattern $(iii) - (iv) - (iii)$. Arnott et. al. (1993b) pointed out so-called "A dynamic traffic equilibrium paradox". The paradox is that capacity expansion in upstream bottleneck may cause increase in total cost. In this subsection, we show that the paradox of capacity expansion does not arise in any cases. First, we calculate the equilibrium commuters' travel costs and total cost of each user group. Next, we show that capacity expansion decreases the sum of total cost of each user group in all cases.

From (10), (18) and (19), the equilibrium user's travel costs of group A (downstream) is

$$C_A = \frac{n_A}{L_T^A - L_T^B} \frac{\beta\gamma}{\beta + \gamma}, \tag{26}$$

and total cost of group A is

$$TC_A = \frac{n_A^2}{L_T^A - L_T^B} \frac{\beta\gamma}{\beta + \gamma}. \tag{27}$$

Therefore, the change of total cost of group A according to capacity expansion in upstream bottleneck is

$$\frac{\partial TC_A}{\partial L_T^B} = \left(\frac{n_A}{L_T^A - L_T^B} \right)^2 \frac{\beta\gamma}{\beta + \gamma}. \tag{28}$$

On the other hand, from (11), (21) and (22), the equilibrium user's travel costs of group B (upstream) is

$$C_B = \frac{n_A}{L_T^B} \frac{\beta\gamma}{\beta + \gamma}, \tag{29}$$

and total cost of group B is

$$TC_B = \frac{n_B^2}{L_T^B} \frac{\beta\gamma}{\beta + \gamma}. \tag{30}$$

Therefore, the change of total cost of group B according to capacity expansion in upstream bottleneck is

$$\frac{\partial TC_B}{\partial L_T^B} = -\left(\frac{n_B}{L_T^B} \right)^2 \frac{\beta\gamma}{\beta + \gamma}. \tag{31}$$

From these results and $\frac{n_A}{L_T^A - L_T^B} < \frac{n_B}{L_T^B}$, we can conclude that for all cases, capacity expansion in upstream bottleneck must decrease the sum of total costs of each group.

Proposition 2 *In two-tandem (Y-shaped) bottleneck case, for any capacity expansion in upstream bottleneck must decrease the sum of total costs of each group.*

4 Charging rule for monocentric cities

4.1 Take the maximum of $\frac{N(r)}{L_T(r)}$ and the minimum capacity at the CBD

First, we will consider the case that maximum $\frac{N(r)}{L_T(r)}$ is taken at r_c (which is edge of the CBD). In this case, by charging at edge of the CBD, we can attain maximization of social surplus and clear the queue. The reason is that charging which depends on $L_T(r_c)$ makes departure rate per time from $[r_f\ ,r_c]$ equal $L_T(r_c)$ and makes no bottleneck other than $L_T(r_c)$. In this case, the toll is

$$\tau(t,r_c) = \begin{cases} \frac{\beta\gamma}{\beta+\gamma}\frac{N(r_c)}{L_T(r_c)} - \beta(t^* - t) & if \quad t \in [t_0(r_c), t^*] \\ \frac{\beta\gamma}{\beta+\gamma}\frac{N(r_c)}{L_T(r_c)} - \gamma(t - t^*) & if \quad t \in [t^*(r_c), t_e] \end{cases} . \tag{32}$$

This toll is the same as in the toll of original ADL model. Hence, this charging brings maximum surplus comparing any chargings at other points. This reason is clear because the bottleneck is unique (from the proposition 1).

Proposition 3 *At the closed monocentric city, the point which attains the maximum traffic volume/capacity and minimum capacity on r locates at r_c, the maximum social welfare is attained by imposing ADL charging at r_c. By imposing this charging, congestion will disappear.*

In many practical cases of Road Pricing, the charging points are often located on bridges or tunnels near the CBD because of technical feasibility. However, we could see that these charging points are selected properly in terms of efficiency also. Because such facilities have almost maximum $\frac{N(r)}{L_T(r)}$ in many cities.

4.2 Take the maximum of $\frac{N(r)}{L_T(r)}$ and the minimum capacity at other than the CBD

Next, we consider the case that maximum $\frac{N(r)}{L_T(r)}$ is taken at other than r_c (i.e., $r_m^1 \in (r_c, r_f]$). And we suppose that second largest $\frac{N(r)}{L_T(r)}$ is attained at the edge of the CBD (i.e., $r_m^2 = r_c$). For the road users located in $r \in (r_m^1, r_f]$ ($N(r_m^1) = \int_{r_m^1}^{r_f} n(x)dx$), we will impose the ADL charging at r_m^1 to make their departure rate $L_T(r_m^1)$. The rush hour interval is $[t_0(r_m^1), t_e(r_m^1)]$ which is maximum for all r.

For $r \in [r_m^2, r_m^1)$, the capacities are strictly larger than $L_T(r_m^1)$ (proof in appendix A) and from the assumption, the maximum of $\frac{N(r)}{L_T(r)}$ in this section is attained at r_c. Therefore we need to consider that the charging rule about $N(r_c) - N(r_m^1)$ on $L_T(r_c) - L_T(r_m^1)$. The rush interval $[t_0(r_c), t_e(r_c)]$ is the proper subset of $[t_0(r_m^1), t_e(r_m^1)]$. Hence, we can consider that the available capacity at r_c is $(L_T(r_c) - L_T(r_m^1))$ for all time in $[t_0(r_c), t_e(r_c)]$ (proof in appendix B). Therefore we can simply apply the ADL charging at two points for each road user group respectively. By this charging, the departure rate of road users locating in $(r_m^1, r_f]$ will be the same of "remaining" road capacity at r_c, i.e., $L_T(r_c) - L_T(r_m^1)$. Regardless of the more than two bottlenecks case, if the r_m^i corresponds with r_c for any i, the same rule can be applied.

Proposition 4 *At the closed monocentric city, the point which attains the maximum traffic volume/capacity on r locates at other than r_c, and the point which attains the second largest traffic volume/capacity and minimum capacity on r locates at r_c. Then, the social welfare* *is maximized by imposing the ADL chargings at r_m^1 and r_c for each section divided by r_m^1. By imposing these chargings, congestion will disappear.*

4.3 Discussions

The case is that $r_m^1 > r_c$, and r_m^i close to r_c asymptotically, however never correspond with r_c. In this case, we can define the charging theoretically, however it is not realistic to charge at infinite points. In this rule, finite times of charging occur only the case that for any r_m^i corresponds with r_c. If not so, we must use arbitrary stopping rule in practice.

Next discussion is that the point which takes second largest $\frac{N(r_m^2)}{L_T(r_m^2)}$ closes to largest $\frac{N(r_m^1)}{L_T(r_m^1)}$ asymptotically. In this case, we can not find the $\frac{N(r_m^2)}{L_T(r_m^2)}$, hence we can not define the charging rules at the $\frac{N(r_m^2)}{L_T(r_m^2)}$ either.

For more proceeding with analysis, we need to relax the assumption of fixed location. In this analysis, we have fixed the distribution of road users on r. That is certain that congestion charging is short run problem, on the other hand, location optimization is long run problem. However location choice of individuals will be strongly affected by transport cost, hence, we need to analyze the relation between ADL charging and location choice. Also, to find the optimal road capacity at each point is remaining problem.

4.4 Summary

In this paper, we consider the relation among commuters' location, road capacity distribution and occurrence of bottlenecks. And also, we examine the relation between road capacity expansion and welfare, and the applicability of the ADL charging for monocentric cities with continuous land. We examine the occurrences condition of bottlenecks under several possible bottleneck locations. The realization patterns of bottlenecks depend on relative sizes (i.e., traffic volume/capacity) of bottlenecks. As we see in proposition 1, if the possible bottleneck location near the CBD has a less traffic volume/capacity, both possible bottleneck locations become bottlenecks. On the other hand, if the possible bottleneck location near the CBD has a much traffic volume/capacity, only this possible bottleneck location becomes a bottleneck. Also we examine that, for any cases, the "Paradox" never occur ; therefore, the capacity expansion of upstream bottleneck always lead the total surplus increase. Under monocentric city, we show that the applicability depends on the location of maximum $\frac{N(r)}{L_T(r)}$. If maximum $\frac{N(r)}{L_T(r)}$ is attained at r_c (which is edge of the CBD), the maximum social welfare is attained by imposing the ADL charging at r_c. By imposing this charging, congestion will disappear. On the other hand, If maximum $\frac{N(r)}{L_T(r)}$ is taken at other than r_c (i.e., $r_m^1 \in (r_c, r_f]$), we need to check the applicability of charging case by case. Each section except $[r_m^1, r_f]$ do not satisfy the compactness, hence, the applicability is limited in such a case; "if there exists the n-th largest $\frac{N(r)}{L_T(r)}$". However, if there exist, we can define a charging rule from r_m^1 to r_m^n. And if the n-th largest $\frac{N(r)}{L_T(r)}$ is taken at r_c, we can define the rule which requires finite charging points. Each commuter pays the ADL toll only one time which depend on $\frac{N(r)}{L_T(r)}$ at the nearest charging point from his departed location. By this charging, social welfare is maximized.

By this analysis, we show that even relatively simple scheme could attain optimum. Therefore, complicate and expensive schemes (such as GPS charging for trip distance) are not necessarily required to attain social optimum on monocentric cities.

Appendix 1

We show the realization patterns of bottlenecks in two possible bottleneck locations case. From the assumptions and FOC, some patterns will never occur. Therefore, we prove the possible patterns.

The case (i) do not satisfy FOC, hence it never occur. Also the patterns which include the case (i) never exist.

Next, we consider patterns including the case (ii). In the case (ii), $D_x^B(t) = 0$ and $\tilde{t}_B = \tilde{t}_A$. Hence, $D_x^B(t) > 0$ never occurs after the case (ii). Therefore, pattern $(ii) - (iii)$ and pattern $(ii) - (iv)$ do not happen.

Then, we examine the patterns including the case (iii) First, there are not t which satisfy the FOC of n_A, hence the case (iii) never occurs solely. Second, it is impossible to move from the case (iii) to the case (ii) before $t_{q'}^B$ ($t \leq t_{q'}^B \implies D_x^B(t) > 0$) and it violates equilibrium condition to move from the case (ii) to the case (iii), therefore pattern $(iii) - (ii)$ and pattern $(iii) - (iv) - (iii) - (ii)$ do not occur. Finally these are impossible to end with the case (iv) and to move from the case (iv) to the case (ii) because $D_x^B(t) > 0$ which arise from the time interval $[t_q^B, t_q^A]$ is kept in a period of the case (iv),.Hence, pattern $(iii) - (iv)$, pattern $(iii) - (iv) - (ii)$, pattern $(iii) - (iv) - (ii) - (iii)$, pattern $(iii) - (iv) - (ii) - (iv)$ and pattern $(iii) - (iv) - (iii) - (iv)$ are not exist.

Finally, the cases starting from the case (iv) do not exist because these contradict $D_x^B(t) > 0$.

Therefore the possible patterns are only pattern (ii) and pattern $(iii) - (iv) - (iii)$.

Appendix 2

We prove that the set of departure times of r_m^{i+1} becomes the proper subset of the set of departure times of r_m^i i.e., $[t_0(r_m^{i+1}), t_e(r_m^{i+1})] \subset [t_0(r_m^i), t_e(r_m^i)]$. First, $\frac{N(r_m^i)}{L_T(r_m^i)} = \frac{\frac{N(r_m^i)}{L_T(r_m^i)}(L_T(r_m^{i+1}) - L_T(r_m^i))}{L_T(r_m^{i+1}) - L_T(r_m^i)} = \frac{\frac{L_T(r_m^{i+1})}{L_T(r_m^i)}N(r_m^i) - N(r_m^i)}{L_T(r_m^{i+1}) - L_T(r_m^i)}$. Then, $\frac{N(r_m^i)}{L_T(r_m^i)} > \frac{N(r_m^{i+1})}{L_T(r_m^{i+1})} \Rightarrow \frac{L_T(r_m^{i+1})}{L_T(r_m^i)}N(r_m^i) > N(r_m^{i+1})$, we can get $\frac{N(r_m^i)}{L_T(r_m^i)} > \frac{N(r_m^{i+1}) - N(r_m^i)}{L_T(r_m^{i+1}) - L_T(r_m^i)}$. All commuters have the same t^*, α, β, γ, we can get $\frac{N(r_m^i)}{L_T(r_m^i)} > \frac{N(r_m^{i+1})}{L_T(r_m^{i+1})} \Rightarrow [t_0(r_m^i), t_e(r_m^i)] \supset [t_0(r_m^{i+1}), t_e(r_m^{i+1})]$.

References

[1] Arnott, R., de Palma, A., Lindsey, R., (1988). “Schedule Delay and Departure Time Decisions with Heterogeneous Commuters” Transportation Research Record 1197, 56-67.

[2] Arnott, R., de Palma, A., Lindsey, R., (1990a). "Economics of a bottleneck" Journal of Urban Economics 27, 111-130.

[3] Arnott, R., de Palma, A., Lindsey, R., (1990b). "Departure time and route choice for routes in parallel" Transportation Research A 25A, 309-318.

[4] Arnott, R., de Palma, A., Lindsey, R., (1993b). "A dynamic traffic equilibrium paradox" Transportation Science 27, 148-160.

[5] Arnott, R., de Palma, A., Lindsey, R., (1998). "Recent development in the bottleneck model" in Road Pricing, Traffic Congestion and the Environment, edited by Button, K. Verhoef, E.

[6] Daniel, T., Gisches, E., Rapoport, A., (2009). "Departure Times in Y-Shaped Traffic Networks with Multiple Bottlenecks" American Economic Review 2009, 99:5, 2149-2176.

[7] Fujita, M., (1989). Urban Economic Theory-Land Use and City Size-. Cambridge University Press.

[8] Kuwahara, M., (1990). "Equilibrium Queuing Patterns at a Two-Tandem Bottleneck during the Morning Peak" Transportation Science 24, 217-229.

[9] Tabuchi, T., (1993). "Bottleneck congestion and modal split" Journal of Urban Economics 34, 414-431.

[10] Verhoef, E., Nijkamp, P., Rietveld, P., (1996). "Second-best congestion pricing :The case of an untolled alternative" Journal of Urban Economics 40, 279-302.

[11] Vickrey, W., (1969). "Congestion Theory and Transport Investment" American Economic Review, 59(2), 251-260.