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Second Best Efficiency of Strategy-proof Allocation Rules and a Characterization of the Uniform Rule with Multiple Commodities

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Abstract

This note studies the design of a strategy-proof resource allocation rule in an environment with perfectly divisible commodities and single-peaked preferences. In this environment, *the uniform rule* has played a central role in the literature. Sprumont(1991) and Ching(1994) show that the uniform rule is the only rule which satisfies *strategy-proofness*, *Pareto efficiency* and some equity conditions when there is only one commodity. However, in the environment with multiple commodities the uniform rule does not satisfy *Pareto efficiency*. We adopt a concept of second best efficiency instead of *Pareto efficiency*. In the main theorem, we give a full characterization of the uniform rule with the second best efficiency in a two person economy. *Journal of Economic Literature* Classification Numbers : D63, D71.

Keywords: Strategy-proofness; Second best efficiency; Generalized uniform rule; single-peaked preferences.

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1 Introduction

One of the most serious problem in designing a resource allocation rule is the trade-off between *strategy-proofness*(each agent cannot be better off by misreporting one's preference) and *Pareto efficiency*(allocations selected by a rule are Pareto efficient: that is, there is no another feasible allocation in which every agent is not worse off and at least one agent is better off). Particularly in the context of the pure exchange economies, it is well-known that rules which satisfy both *strategy-proofness* and *Pareto efficiency* admit the unsymmetrical treatment of agents, typically the existence of the dictator (See Hurwicz (1972), Zhou (1991), Serizawa (2002) and Serizawa and Weymark (2003)).

However, very different result is known in an environment with *single-peaked preferences*. In his celebrated paper, Sprumont (1991) shows that the uniform rule is the only rule which satisfies *anonymity* (rules do not depend on names of agents) (or *envy-freeness*(no agent prefers others' consumption to one's own consumption)), *strategy-proofness* and *Pareto efficiency* in the economy with only *one* commodity and *single-peaked preferences*. Roughly speaking, the uniform rule is the rule that allocates the same consumption for each individual, but if the sum of individuals' peaks is greater (smaller) than the total amount of the good supplied, then individuals whose peaks are small(large) enough receive their own peaks and individuals whose peaks are large(small) receive the equal division of the residual. That is, individuals whose peaks are small enough have the priority if there is excess demand, and individuals whose peaks are large enough have the priority if there is excess supply under the uniform rule. In Sprumont's environment, the *strategy-proofness* and the *Pareto efficiency* do not imply the unsymmetrical treatment of agents unlike the pure exchange economy (See Sprumont (1991) and Ching (1994)).

Single-peaked preferences and the uniform rule can naturally be extended to m commodity environments. The uniform rule in the environment with more than one commodities, which is referred to as *the generalized uniform rule*, allocates commodities commodity by commodity on the basis of the uniform rule in the one commodity environment. It may be a natural question : Does the generalized uniform rule satisfy *strategy-proofness*, *Pareto efficiency* and many equity conditions like one-dimensional uniform rule? The answer for the *strategy-proofness* and equity conditions is positive, but, unfortunately enough, the answer for the *Pareto efficiency* is negative; the generalized uniform rule is *not* Pareto efficient. Moreover, Amorós (2002) pointed out that *strategy-proofness* and *Pareto efficiency* together result in the dictatorship under the single-peaked preference environment with more than one commodities when the number of individual is two. Even under the single-peaked preference environment, if we suppose that the economy has several commodities, then the conflict between *strategy-proofness* and *Pareto efficiency* arises. We study the design of a rule in this environment. Particularly, we consider rules which satisfies strategy-proofness, some equity conditions and a notion of second best efficiency instead of Pareto efficiency. We give a full characterization of *the generalized uniform rule* when there are only two agents.

Regardless of the number of commodities, the (generalized) uniform rule satisfies a solidarity condition called *same-sidedness* which is a necessary condition for *Pareto efficiency*. The *same-sidedness* is equivalent to the *Pareto efficiency* only if there is only one commodity, and this is the key to Sprumont's successful characterization. However the condition in the environment with more than one commodities is strictly weaker than *Pareto efficiency*. We discuss the condition in section 4 in detail.

Amorós (2002) characterized the generalized uniform rule by *strong symmetry*(agents who have the same preference recieve the same allotment) (or *envy-freeness*), *strategy-proofness* and *same-sidedness* in 2 person m goods economy. Morimoto, Ching and Serizawa (2008) characterized the rule by *symmetry*(agents who have the same preference recieve indifferent allotments for them), *strategy-proofness*, *respect for unanimity*(if the sum of peaks coincides with the social endowment, then each agent recieves one's own peak) and *non-bossiness*[1] in n person m goods economy. *Respect for unanimity* is a necessary for *same-sidedness*, and it is very weaker than *same-sidedness* in general. However it is equivelent to *same-sidedness* under *strategy-proofness* and *non-bossiness*. Hence Morimoto, Ching and Serizawa's (2008) characterization is in line with Amorós' characterization with *same-sidednss*.

Although the generalized uniform rule is not Pareto efficient, it is reasonable to expect that the generalized uniform rule has some efficiency properties relative to other strategy-proof rules. Indeed, Sasaki (2003) shows that the generalized uniform rule satisfies a notion of second best efficiency. More precisely, he showed that there exists no strategy-proof rule which Pareto dominates the generalized uniform rule for all preference profiles even when the number of commodities is more than one but the number of individuals is two. His result indicates that the generalized uniform rule is still a prominent presence with respect to efficiency among the strategy-proof rules in two person m commodities environment.

In this note, we introduce a second best efficiency condition which is weaker than Sasaki's, and characterize the generalized uniform rule with the condition in two person economies. This note proceeds as follows: in section 2 we introduce our model and axioms formally. Section 3 describes our main result. In section 4 we compare our result with literatures'. Section 5 concludes.

2 Setup

2.1 Notations

Let $N = \{1, 2\}$ be the set of individuals. Let $M = \{1, \cdots, m\}$ represent the set of commodities. All commodities are perfectly divisible. The bundle $\Omega = (\Omega_1, \cdots, \Omega_m) \in \mathbb{R}^m_{++}$ denotes the endowment of commodities in this economy. [2] $B = \{\mathbf{x} = (x^1, x^2) \in (\mathbb{R}^m_+)^2 \mid x^1 + x^2 = \Omega\}$ denotes the set of feasible allocations. We do not allow free disposal. $\mathcal{R}$ denotes the set of complete, transitive and continuous binary relations on $\prod_{j=1}^m [0, \Omega_j]$.[3]

[1] If a change of an agent's preference does not change the agent's allotment, then it does not change other agents' allotment

[2] $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative and positive real numbers, respectively.

[3] For each $R \in \mathcal{R}$, $P(R)$ and $I(R)$ denote the asymmetric part of R and the symmetric part of R, respectively.

Definition. $R \in \mathcal{R}$ is single-peaked if there exists $p(R) \in \prod_{j=1}^{m}[0,\Omega_j]$ such that for all $x, x' \in \prod_{j=1}^{m}[0,\Omega_j](x \neq x')$,

$$\left[\forall j \in M : x'_j \leq x_j \leq p_j(R) \vee p_j(R) \leq x_j \leq x'_j\right] \Rightarrow x\, P(R)\, x'.$$

Let $\mathbb{S} = \{R \in \mathcal{R} \mid R \text{ is single-peaked}\}$. $\mathbb{S}^N$ denotes the set of preference profiles. For each preference profile $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, $\mathbf{R}^{-1} = R^2$ and $\mathbf{R}^{-2} = R^1$.

$\tau(R, Y)$ denotes the best consumptions on $Y \subseteq \prod_{j=1}^{m}[0,\Omega_j]$ with respect to $R \in \mathbb{S}$. That is, $\tau(R, Y) = \{x \in Y \mid \forall y \in Y : xRy\}$. In our setting, each individual has a continuous preference on compact consumption set $\prod_{j=1}^{m}[0,\Omega_j]$. Hence, if $Y \subseteq \prod_{j=1}^{m}[0,\Omega_j]$ is closed, then $\tau(R, Y)$ is not empty.

A function from $\mathbb{S}^N$ to B is called *a rule*. The symbol $B^{f^i_R}$ denotes the range of the ith component function of f when we fix the others' preference R. That is, given a rule f, for all $R \in \mathbb{S}$, $B^{f^1_R} = f^1(\mathbb{S} \times \{R\})$ and $B^{f^2_R} = f^2(\{R\} \times \mathbb{S})$. We refer $B^{f^i_R}$ as *the option set of individual i with respect to* (f, R).

The purpose of this note is to characterize the following rule which is known as *the generalized uniform rule*.

Definition. The generalized uniform rule $U : \mathbb{S}^N \to B$ is the rule defined by the following; for all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, all $j \in M$, all $i \in N$,

$$U^i_j(\mathbf{R}) = \begin{cases} \min\{p_j(R^i), \lambda_j(\mathbf{R})\} & \text{if } p_j(R^1) + p_j(R^2) \geq \Omega_j, \\ \max\{p_j(R^i), \mu_j(\mathbf{R})\} & \text{if } p_j(R^1) + p_j(R^2) \leq \Omega_j, \end{cases}$$

where $\lambda_j(\mathbf{R}) = \Omega_j - \min\{p_j(R^1), p_j(R^2)\}$ and $\mu_j(\mathbf{R}) = \Omega_j - \max\{p_j(R^1), p_j(R^2)\}$.

2.2 Axioms

The following axiom requires that one cannot be better off by misreporting one's preference.

Strategy-proofness (SP) : For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$ and all $\hat{R} \in \mathbb{S}$, $f^1(\mathbf{R})\, R^1\, f^1(\hat{R}, R^2)$ and $f^2(\mathbf{R}) R^2 f^2(R^1, \hat{R})$.

Γ_{SP} denotes the set of strategy-proof rules. The following axiom reflects the idea that everyone should not be worse off than under the equal division rule.

Egalitarian rationality (ER) [4]: For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$ and all $i \in N$, $f^i(\mathbf{R})\, R^i\, \frac{\Omega}{2}$.

The following axiom requires that the rules should be "simple" in the sense a rule depends only on the each individual's peak of preference.

[4]This axiom is sometimes referred to as *the equal division lower bound*.

Peak-onliness : For all $\mathbf{R} = (R^1, R^2), \tilde{\mathbf{R}} = (\tilde{R}^1, \tilde{R}^2) \in \mathbb{S}^N$, if $p(R^1) = p(\tilde{R}^1)$ and $p(R^2) = p(\tilde{R}^2)$, then $f(\mathbf{R}) = f(\tilde{\mathbf{R}})$.

The next axiom requires that for each individual, his allotment can not change without changing his peak under the condition other individuals' preferences are unchanged. Note that this axiom is weaker than peak-onliness in general. But in our 2 person setup, peak-onliness and weak peak-onliness are equivalent.

Weak peak-onliness (WP) : For all $\mathbf{R} = (R^1, R^2), \forall \tilde{R}^1, \tilde{R}^2 \in \mathbb{S}$, if $p(R^1) = p(\tilde{R}^1)$, then $f^1(\mathbf{R}) = f^1(\tilde{R}^1, R^2)$ and if $p(R^2) = p(\tilde{R}^2)$, then $f^2(\mathbf{R}) = f^2(R^1, \tilde{R}^2)$.

To introduce our second best efficiency conditions, we need to prepare the next relation on Γ_{SP}.

Definition. A binary relation dom $\subseteq \Gamma_{\mathrm{SP}} \times \Gamma_{\mathrm{SP}}$ is defined as follows; for all $f, g \in \Gamma_{\mathrm{SP}}$

$$f \text{ dom } g \iff \forall \mathbf{R} = (R^1, R^2) \in \mathbb{S}^N, \forall i \in N : f^i(\mathbf{R})\, R^i\, g^i(\mathbf{R}).$$[5]

Our second best efficiency conditions are following two. Note that the latter is logically stronger than the former.

Weak second best efficiency among strategy-proof rules (WSESP) : $f \in \Gamma_{\mathrm{SP}}$ and for all $g \in \Gamma_{\mathrm{SP}}$, if g dom f, then f dom g.

Strong second best efficiency among strategy-proof rules (SSESP) : $f \in \Gamma_{\mathrm{SP}}$ and for all $g \in \Gamma_{\mathrm{SP}}$, if g dom f, then $f = g$.

Sasaki (2003) first introduced *SSESP* and showed that the generalized uniform rule satisfied *SSESP*.

3 Main Result

We now state our main result. The proof is given in Appendix A.

Theorem 1. *The generalized uniform rule is the only rule which satisfies the egalitarian rationality, the weak peak-onliness and the weak second best efficiency among strategy-proof rules.*

The following examples show that Theorem 1 is a tight characterization. That is, the lack of one of three axioms leads to multiplicity of rules.

[5] Note that dom is reflexive and transitive. Hence it is a preorder on Γ_{SP}. But it is not an order on Γ_{SP} in general.

Example 1. *An example of rules which satisfy both ER and WP, but do not satisfy WSESP:* Let E be the equal division rule defined as follows; for all $\mathbf{R} \in \mathbb{S}^N$, $E(\mathbf{R}) = (\frac{\Omega}{2}, \frac{\Omega}{2})$. Obviously E satisfies *ER* and *WP* but E does not satisfy *WSESP* because U dom E and not E dom U.

Example 2. *An example of a rules which satisfy both WP and WSESP, but do not satisfy ER:* Let $D^{(1)}$ be the priority rule in which individual 1 has the priority defined as follows; for all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, $D^{(1)}(\mathbf{R}) = (p(R^1), \Omega - p(R^1))$. Since $D^{(1)}$ is *SP* and Pareto efficient, it must be *WSESP*. It is also clear that $D^{(1)}$ satisfies *WP*. However, clearly $D^{(1)}$ does not satisfy *ER*.

Example 3. *An example of a rule which satisfies both ER and WSESP, but does not satisfies WP:* Let f be the rule defined as follows. For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$,

$$f(\mathbf{R}) = \begin{cases} (\Omega, 0) & \text{if } \Omega\, R^1 \frac{\Omega}{2} \text{ and } 0\, R^2 \frac{\Omega}{2}, \\ (\frac{\Omega}{2}, \frac{\Omega}{2}) & \text{otherwise.} \end{cases}$$ [6]

Obviously f satisfies *SP* and *ER*.

First we show that U does not dominate f. By Lemma 3, there exists $\tilde{R}^1, \tilde{R}^2 \in \mathbb{S}$ such that $p(\tilde{R}^1) = p(\tilde{R}^2) = (\Omega_1, \cdots, \Omega_{m-1}, 0)$ and $\Omega\ P(\tilde{R}^1)\frac{\Omega}{2}$ and $0\ P(\tilde{R}^2)\frac{\Omega}{2}$. Then $f(\tilde{R}^1, \tilde{R}^2) = (\Omega, 0)$ by Lemma 1. Since $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$, U does not dominate f.

By Theorem 3.1 in Anno(2008), there exists $f_0 \in \Gamma_{\text{SP}}$ such that f_0 is *WSESP* and f_0 dom f. Since f satisfies *ER* and dom is transitive, f_0 satisfies *ER*. $f_0^1(\tilde{R}^1, \tilde{R}^2)\tilde{R}^1 \Omega\ P(\tilde{R}^1)\frac{\Omega}{2}$ and $f_0^2(\tilde{R}^1, \tilde{R}^2)\tilde{R}^2\ 0\ P(\tilde{R}^2)\frac{\Omega}{2}$ because f_0 dom f. Since $U(\tilde{R}^1, \tilde{R}^2) = (\frac{\Omega}{2}, \frac{\Omega}{2})$, $f_0 \neq U$. This means that f_0 does not satisfy *WP*, because if f_0 satisfies *WP*, then by Theorem 1 $f_0 = U$, a contradiction.

By Example 1,2 and 3, we have shown that *axioms in Theorem 1 are mutually independent*.

4 Discussion

As was mentioned in the introduction, the generalized uniform rule is not Pareto efficient, but satisfies logically weaker conditions, namely *same-sidedness* and *respect for unanimity*.

Pareto efficiency : For all $g \in \Gamma$, if g dom f, then f dom g. [7]

Same-sidedness : For all $\mathbf{R} \in \mathbb{S}^N$, and all $j \in M$, (1) if $\sum_{i \in N} p_j(R^i) \geq \Omega_j$, then $f_j^i(\mathbf{R}) \leq p_j(R^i)$ for all $i \in N$, and (2) if $\sum_{i \in N} p_j(R^i) \leq \Omega_j$, then $p_j(R^i) \leq f_j^i(\mathbf{R})$ for all $i \in N$. [8]

[6] This kind of rules are first introduced by Hiroo Sasaki.

[7] Note that this is equivalent to the following statement. For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, there is no $\mathbf{x} = (x^1, x^2) \in B$ such that (1) $x^i R^i f^i(\mathbf{R})$ for all $i \in N$, and (2) $x^{i_0} P^{i_0} f^{i_0}(\mathbf{R})$ for some $i_0 \in N$.

[8] It is well-known that *same-sidedness* is equivalent to *Pareto efficiency* if $m = 1$. See Sprumont (1991).

Respect for unanimity : For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, if $\sum_{i\in N} p(R^i) = \Omega$, then $f(\mathbf{R}) = (p(R^1), p(R^2))$.

Note that *respect for unanimity* is much weaker than *same-sidedness* in general, but in our setup they are equivalent under *SP* (See Lemma 1 in Morimoto, Ching and Serizawa (2008)). Amorós (2002) pointed out the following.

Proposition 1 (Amorós (2002)). Suppose that $f \in \Gamma_{\mathrm{SP}} \setminus \{D^{(1)}, D^{(2)}\}$. Then f does not satisfy *Pareto efficiency*.

According to Proposition 1, if we hope to design a non-dictatorial rule which satisfies *SP*, then we must give up *Pareto efficiency*. Amorós (2002) and Morimoto, Ching and Serizawa (2008) chose *same-sidedness* or *respect for unanimity* instead of *Pareto efficiency*. Note that Proposition 2 is logically implied by Proposition 3.

Proposition 2 (Amorós (2002)). The generalized uniform rule is the only rule which satisfies *SP*, *same-sidedness* and *strong symmetry*. [9]

Proposition 3 (Morimoto, Ching and Serizawa (2008)). The generalized uniform rule is the only rule which satisfies *SP*, *respect for unanimity* and *symmetry*. [10] [11]

Note that equity conditions *ER, symmetry* and *strong symmetry* imply non-dictatorship. As was pointed out in Proposition 1, if we hope to design a *SP* and non-dictatorial rule, then we must give up *Pareto efficiency*. That is, we must admit some profiles which are not corresponded to a Pareto efficient allocation. Hence, they can be interpreted as the "cost" of *SP* and non-dictatorship.

It is hard for us to decide profiles which is corresponded to non Pareto efficient allocation. *Same-sidedness* and *Respect for unanimity* are restrictive in the sense that they restrict profiles in which non Pareto efficiency arise to profiles in which $\sum_{i\in N} p(R^i) \neq \Omega$. But this is less attractive because there is no theoretical reason we treat profiles in which $\sum_{i\in N} p(R^i) = \Omega$ as exceptions.

It is easy to check that if we have a rule which does not satisfy *respect for unanimity* (*Same-sidedness*), then we can construct a more efficient rule. Formally, if a rule f does not satisfies *respect for unanimity*, then we have at least one preference profile $\mathbf{R}' = (R'^1, R'^2) \in \mathbb{S}^N$ such that $\sum_{i\in N} p(R'^i) = \Omega$ but $f(\mathbf{R}') \neq (p(R'^1), p(R'^2))$. And obviously we can constract more efficient rule f' by defining $f'(\mathbf{R}') = (p(R'^1), p(R'^2))$ and $f'(\mathbf{R}) = f(\mathbf{R})$ for all $\mathbf{R} \neq \mathbf{R}'$. Amorós's justification for *Same-sidedness* is to exclude this type of construction.[12] Note that in the example above *SP is not inherited*

[9] *Strong symmetry* : For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, if $R^1 = R^2$, then $f^1(\mathbf{R}) = f^2(\mathbf{R})$.

[10] *Symmetry* : For all $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$, if $R^1 = R^2$, then $f^1(\mathbf{R}) I(R^1) f^2(\mathbf{R})$.

[11] Morimoto, Ching and Serizawa (2008) extremly extend Amorós' (2002) characterization result. They showed that the generalized uniform rule is the only rule satisfying *SP, respect for unanimity*, *symmetry* and *non-bossiness in n person m commodity economy*. *Non-bossiness* requires that a change of an individual's preference under the condition his allotment is unchanged does not change the allotment for others. *Non-bossiness* is trivially satisfied in our setup with only two individuals and the feasibility condition. Hence, **Proposition 3** is a special case of Morimoto, Ching and Serizawa's result.

[12] *Same-sidedness* is referred to as *Condition E* in Amorós (2002).

by f' in general when f is *SP*.

Imposing *Respect for unanimity* (*Same-sidedness*) with *SP* excludes the *SP* rules which can be dominated by another (possibly not *SP*) rule. However, in this note, *WSESP* excludes only the *SP* rules which can be dominated by another *SP* rule. This gives us an intuition that *SP* and *respect for unanimity* together imply our second best efficiency condition. Theorem 2 (1) certifies this intuition to be true. In fact, *respect for unanimity* with *SP* implies *SSESP*. Note that Example 3 also showed that *WSESP* does not imply *respect for unanimity*. [13] The proof of Theorem 2 is given in Appendix B.

Theorem 2.
(1) If f satisfies *respect for unanimity* and *SP*, then f satisfies *SSESP*.
(2) If f satisfies *WP* and *symmetry*, then f satisfies *strong symmetry*.
(3) If f satisfies *WP*, *symmetry* and *SP*, then f satisfies *ER*.

(1) in Theorem 2 shows a sufficient condition for *SSESP* and note that the generalized uniform rule is a rule which satisfies the *respect for unanimity* and *SP*. That is, this result strengthens Sasaki's (2003).

Note also that Theorem 1 and 2 together[14] *imply Proposition 2 and Proposition 3.*

5 Concluding Remarks

In this note, we have shown that the generalized uniform rule is the only rule that satisfies *ER*, *WP* and *WSESP* in two person *m* good economy with single-peaked preferences. Furthermore, we also showed that axioms we employed are independent.

In the main theorem, we employed *WSESP*, but not just *SP*. In Appendix C, examples of rules which satisfy *SP, ER* and *WP* other than the generalized uniform rule and the equal division rule are provided. It shows that there are uncountably many rules which satisfy *SP, ER* and *WP*. This shows that *WSESP*, not just *SP*, is crucial in our characterization.

Whether our results can be extended to models with more than 2 individuals is still open. We hope that some of our techniques help to solve the problem.

Appendix A : Proof of Theorem 1

Note that the assumption that there are only 2 individuals does not matter in Lemma 1,2,3 and 4.

Lemma 1. Let f be a rule.

$$f \in \Gamma_{\mathrm{SP}} \Leftrightarrow \forall \mathbf{R} \in \mathbb{S}^N, \forall i \in N : f^i(\mathbf{R}) \in \tau(R^i, B^{f^i_{\mathbf{R}^{-i}}}).$$

[13] If so, the rule must be *WP* by Lemma 3 in Amorós (2002).
[14] See also Lemma 3 in Amorós (2002) and Lemma 1 in Morimoto, Ching and Serizawa (2008)

Proof. Obvious.[15]

Lemma 2. Let $f, g \in \Gamma_{\text{SP}}$.

$$f \text{ dom } g \Leftrightarrow \forall i \in N, \forall R \in \mathbb{S} : B^{g^i_R} \subseteq B^{f^i_R}.$$

Proof. ($\Rightarrow$) Let $i \in N, R \in \mathbb{S}$. Without loss of generality, we assume $i = 1$. Pick $x \in B^{g^1_R}$ arbitrarily. We can take $R^1 \in \mathbb{S}$ such that $\tau(R^1, \prod_{j=1}^m [0, \Omega_j]) = \{x\}$. By Lemma 1, $g^1(R^1, R) = x$. Since f dom g, $f^1(R^1, R)\ R^1\ g^1(R^1, R)$. Hence $f^1(R^1, R) = x$. This means $x \in B^{f^1_R}$.
($\Leftarrow$) Let $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$. By Lemma 1, $f^i(\mathbf{R}) \in \tau(R^i, B^{f^i_{\mathbf{R}^{-i}}})$ and $g^i(\mathbf{R}) \in \tau(R^i, B^{g^i_{\mathbf{R}^{-i}}})$ for all $i \in N$. Since $B^{g^i_{\mathbf{R}^{-i}}} \subseteq B^{f^i_{\mathbf{R}^{-i}}}$, $f^i(\mathbf{R})\ R^i\ g^i(\mathbf{R})$. □

Lemma 3. (Amorós (2002)) If $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j]$ satisfy

$$\left[\exists j \in M \text{ s.t. } |x^*_j - x''_j| < |x^*_j - x'_j|\right] \vee \left[\exists j \in M \text{ s.t. } (x^*_j - x''_j)(x^*_j - x'_j) < 0\right],$$

then there exists $R \in \mathbb{S}$ such that $p(R) = x^*$ and $x'' P(R) x'$.

For the purpose of reference, we prepare an equivalent expression of Lemma 3.

Lemma 4. If $x^*, x', x'' \in \prod_{j=1}^m [0, \Omega_j] (x' \neq x'')$ satisfy

$$\neg[\forall j \in M : x^*_j \leq x'_j \leq x''_j \vee x''_j \leq x'_j \leq x^*_j],$$

then there exists $R \in \mathbb{S}$ such that $p(R) = x^*$ and $x'' P(R) x'$.

Lemma 5. Suppose that f satisfies *SP*, *ER* and *WP*. Then U dom f.

Proof. We show that $B^{f^i_R} \subseteq B^{U^i_R}$ for all $i \in N$ and all $R \in \mathbb{S}$. Then we obtain the conclusion by Lemma 2. Without loss of generality, suppose that $i = 1$. Let $R^2 \in \mathbb{S}$. For all $Y \subseteq \prod_{j=1}^m [0, \Omega_j]$, define $sym(Y) = \{y \in \prod_{j=1}^m [0, \Omega_j] \mid \exists x \in Y \text{ s.t. } y = \Omega - x\}$. Then *ER* and the feasibility condition imply that $B^{f^1_{R^2}} \subseteq sym(\text{UC}(R^2, \frac{\Omega}{2}))$.[16]

Step 1. $\forall \tilde{R}^2 \in \mathbb{S} : \left[p(\tilde{R}^2) = p(R^2) \text{ and } \text{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \text{UC}(R^2, \frac{\Omega}{2}) \Rightarrow B^{f^1_{R^2}} \subseteq sym(\text{UC}(\tilde{R}^2, \frac{\Omega}{2}))\right]$.

Suppose not. We have $\tilde{R}^2 \in \mathbb{S}$ such that $p(\tilde{R}^2) = p(R^2)$, $\text{UC}(\tilde{R}^2, \frac{\Omega}{2}) \subseteq \text{UC}(R^2, \frac{\Omega}{2})$ and $B^{f^1_{R^2}} \not\subseteq sym(\text{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Then there exists a consumption bundle x such that $x \in B^{f^1_{R^2}}$ and $x \notin sym(\text{UC}(\tilde{R}^2, \frac{\Omega}{2}))$. Obviously we can take $R_x \in \mathbb{S}$ such that $p(R_x) = x$. By Lemma 1, $f(R_x, R^2) = (x, \Omega - x)$. Since $B^{f^1_{\tilde{R}^2}} \subseteq sym(\text{UC}(\tilde{R}^2, \frac{\Omega}{2}))$, $x \notin B^{f^1_{\tilde{R}^2}}$. Hence $f(R_x, \tilde{R}^2) \neq (x, \Omega - x)$. But this contradicts *WP*.

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[15] If $f \in \Gamma_{\text{SP}}$, then $B^{f^i_R}$ is closed set in $\prod_{j=1}^m [0, \Omega_j]$ for all $i \in N$, $R \in \mathbb{S}$ in our setup. Hence $\tau(R^i, B^{f^i_{\mathbf{R}^{-i}}})$ is not empty for all $\mathbf{R} \in \mathbb{S}^N$ and $i \in N$. See Barberà and Peleg (1990), Le Breton and Weymark (1999).

[16] For each $R \in \mathcal{R}$, and each $x \in \prod_{j=1}^m [0, \Omega_j]$, $\text{UC}(R, x)$ denotes the upper contour set of x with respect to R.

Step 2. $\forall x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}} : \exists \tilde{R}^2 \in \mathbb{S}$ s.t. $p(\tilde{R}^2) = p(R^2)$ and $\Omega - x \notin \mathrm{UC}(\tilde{R}^2, \frac{\Omega}{2})$.

Note that, for each $j \in M$, define

$$a_j = \begin{cases} \Omega_j - p_j(R^2) & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \frac{\Omega_j}{2} & \text{otherwise,} \end{cases} \quad b_j = \begin{cases} \frac{\Omega_j}{2} & \text{if } \frac{\Omega_j}{2} \leq p_j(R^2), \\ \Omega_j - p_j(R^2) & \text{otherwise,} \end{cases}$$

then $B^{U^1_{R^2}} = \prod_{j=1}^m [a_j, b_j]$. Hence $sym(B^{U^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$.

Fix $x \in sym(\mathrm{UC}(R^2, \frac{\Omega}{2}))\backslash B^{U^1_{R^2}}$ arbitrarily. Note that $x \neq \frac{\Omega}{2}$ because $\frac{\Omega}{2} \in B^{U^1_{R^2}}$. Hence $\Omega - x \neq \frac{\Omega}{2}$. We show the following by contradiction.

$$(*) \quad \neg\Big[\forall j \in M : \frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2) \vee p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2}\Big].$$

Suppose not. Then for all $j \in M$, if $\frac{\Omega_j}{2} \leq \Omega_j - x_j \leq p_j(R^2)$, then $\Omega_j - p_j(R^2) \leq x_j$ and $x_j \leq \frac{\Omega_j}{2}$. This is equivalent to $x_j \in [a_j, b_j]$. Similarly we can show that for all $j \in M$, if $p_j(R^2) \leq \Omega_j - x_j \leq \frac{\Omega_j}{2}$, then $x_j \in [a_j, b_j]$. Hence we have shown that $x \in B^{U^1_{R^2}}$, a contradiction. We have obtained $(*)$.

By Lemma 4, there exists $\tilde{R}^2 \in \mathbb{S}$ such that

$$p(\tilde{R}^2) = p(R^2) \text{ and } \frac{\Omega}{2} P(\tilde{R}^2)(\Omega - x). \ \square$$

Proof of Theorem 1. Obviously U satisfies *ER* and *WP*. As is mentioned in section 2.2, Sasaki(2003) showed that U satisfies *SSESP*. Next, we show the converse. Suppose that f satisfies *WSESP*, *WP* and *ER*. By Lemma 5, U dom f. Since f satisfies *WSESP*, f dom U. Since U satisfies *SSESP*, $f = U$. □

Appendix B : Proof of Theorem 2

Before we prove Theorem 2, we show the following lemma about the shape of the option set. Note that Lemma 6 holds in the environment with more than two individuals.

Lemma 6. Suppose that f satisfies *SP* and *WP*. Then

$$\forall i \in N, \forall R \in \mathbb{S}, \forall j \in M, \exists a_j, b_j \in [0, \Omega_j] \text{ s.t. } B^{f^i_R} = \prod_{j=1}^m [a_j, b_j].$$

Proof. Fix $i \in N$ and $R \in \mathbb{S}$ arbitrarily. Without loss of generality, suppose that $i = 1$. Let $R^2 = R$. Proof is done by two steps.

Step 1. $B^{f^1_{R^2}}$ is convex.

To this end, suppose the contrary. That is, we assume

$$\exists \hat{v}, \hat{w} \in B^{f^1_{R^2}}, \exists \lambda \in (0,1) \text{ s.t. } \lambda\hat{v} + (1-\lambda)\hat{w} \notin B^{f^1_{R^2}}.$$

Let $seg[\hat{v},\hat{w}] = \{x \in \prod_{j=1}^{m}[0,\Omega_j] \mid \exists \mu \in [0,1] \text{ s.t. } x = \mu\hat{v} + (1-\mu)\hat{w}\}$. Now we consider the space $B^{f^1_{R^2}} \cap seg[\hat{v},\hat{w}]$. If this space has infinite connected components, then $B^{f^1_{R^2}} \cap seg[\hat{v},\hat{w}]$ has open cover which contains no finite subcover in $seg[\hat{v},\hat{w}]$. But this contradicts that $B^{f^1_{R^2}} \cap seg[\hat{v},\hat{w}]$ is compact. Hence this space has at most finite connected components. This results in

$$\exists v, w \in B^{f^1_{R^2}}, \forall \lambda \in (0,1) \text{ s.t. } \lambda v + (1-\lambda)w \notin B^{f^1_{R^2}}.$$

Let $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w$ and let $\tilde{R} \in \mathbb{S}$ be a preference which satisfies $p(\tilde{R}) = \tilde{x}$.

Case 1. $f^1(\tilde{R},R^2) = v \vee f^1(\tilde{R},R^2) = w$

We prepare a new notation. For each $\mathbf{d} = (d_1,\cdots,d_m) \in \prod_{j=1}^{m}\{e_j,-e_j\}$ and each $y \in \prod_{j=1}^{m}[0,\Omega_j]$, $E(y,\mathbf{d}) = \{z \in \prod_{j=1}^{m}[0,\Omega_j] \mid \exists \gamma_1,\cdots,\gamma_m \in \mathbb{R}_+ \text{ s.t. } z = y + \sum_{j=1}^{m}\gamma_j d_j\}$, where e_j denotes the m-dimensional vector in which jth coordinate is 1 and other coordinates are 0. Without loss of generality, we may assume $f^1(\tilde{R},R^2) = v$. Suppose that for $\mathbf{d} = (d_1,\cdots,d_m), \mathbf{d}' = (d'_1,\cdots,d'_m) \in \prod_{j=1}^{m}\{e_j,-e_j\}$, $v \in E(p(\tilde{R}),\mathbf{d})$ and $w \in E(p(\tilde{R}),\mathbf{d}')$. Suppose also that $\mathbf{d},\mathbf{d}'$ satisfy that

$$(**) \quad \forall j \in M : \left[v \in E(p(\tilde{R}),(-d_j,\mathbf{d}_{-j})) \ \vee \ w \in E(p(\tilde{R}),(-d'_j,\mathbf{d}'_{-j})) \Rightarrow d_j = d'_j\right],$$

where $(-d_j,\mathbf{d}_{-j}) = (d_1,\cdots,d_{j-1},-d_j,d_{j+1},\cdots,d_m)$ and $(-d'_j,\mathbf{d}'_{-j})$ is defined in the same manner. Obviously there exists $j' \in M$ such that $d_{j'} = -d_{j'}$. Since $v_{j'} < \tilde{x}_{j'} < w_{j'}$ or $w_{j'} < \tilde{x}_{j'} < v_{j'}$, by Lemma 3, there exists $\hat{R}^1 \in \mathbb{S}$ such that $p(\hat{R}^1) = \tilde{x}$ and $wP(\hat{R}^1)v$. Hence $f^1(\hat{R}^1,R^2) \neq v$ by Lemma 1. But this contradicts weak peak-onliness of f.

Case 2. $f^1(\tilde{R},R^2) \neq v \wedge f^1(\tilde{R},R^2) \neq w$

Let $c = f^1(\tilde{R},R^2)$. If

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - v_j| < |\tilde{x} - c_j| \ \vee \ (\tilde{x} - v_j)(\tilde{x} - c_j) < 0]$$

or

$$[\exists j \in M \text{ s.t. } |\tilde{x}_j - w_j| < |\tilde{x} - c_j| \ \vee \ (\tilde{x} - w_j)(\tilde{x} - c_j) < 0],$$

then by Lemma 3,

$$\left[\exists \tilde{R}_v \in \mathbb{S} \text{ s.t. } p(\tilde{R}_v) = \tilde{x} \wedge vP(\tilde{R}_v)c\right] \vee \left[\exists \tilde{R}_w \in \mathbb{S} \text{ s.t. } p(\tilde{R}_w) = \tilde{x} \wedge wP(\tilde{R}_b)c\right].$$

This contradicts the fact that $c = f^1(\tilde{R},R^2)$ and f is weakly peak-only. Hence

$$[\forall j \in M \ : \ |\tilde{x}_j - v_j| \geq |\tilde{x} - c_j| \ \wedge \ (\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0]$$

and

$$[\forall j \in M \ : \ |\tilde{x}_j - w_j| \geq |\tilde{x} - c_j| \ \wedge \ (\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0].$$

Suppose that $d, d' \in \prod_{j=1}^{m}\{e_j,-e_j\}$ satisfy $v \in E(p(\tilde{R}),\mathbf{d})$, $w \in E(p(\tilde{R}),\mathbf{d}')$ and the condition $(**)$ in *Case 1*. Fix $j \in M$ arbitrarily. If $d_j = -d'_j$, then $c_j = \tilde{x}_j$ because $(\tilde{x} - v_j)(\tilde{x} - c_j) \geq 0$ and $(\tilde{x} - w_j)(\tilde{x} - c_j) \geq 0$. If $d_j = d'_j$, then v_j and w_j can be represented $v_j = \tilde{x}_j + \lambda d_j$ and $w_j = \tilde{x}_j + \lambda' d_j$ for some λ, λ'. Since $\tilde{x} = \frac{1}{2}v + \frac{1}{2}w$, $\lambda = \lambda'$. Hence $v_j = w_j = \tilde{x}_j$. Since $|\tilde{x}_j - v_j| \geq |\tilde{x} - c_j|$, $|\tilde{x} - c_j| = 0$. Hence $c_j = \tilde{x}_j$. We have shown that $c = \tilde{x}$. $\tilde{x} \notin B^{f^1_{R^2}}$ and $c \in B^{f^1_{R^2}}$, a contradiction.

Step 2. $\forall j \in M, \exists a_j, b_j \in [0, \Omega_j]$ s.t. $B^{f^1_{R^2}} = \prod_{j=1}^m [a_j, b_j]$.

For each $j \in M$, let Pr_j denote the projection with respect to jth coordinate. Since Pr_j is continuous and $B^{f^1_{R^2}}$ is compact, $\mathrm{Pr}_j(B^{f^1_{R^2}}) \subseteq [0, \Omega_j]$ is compact. Let $a_j = \min \mathrm{Pr}_j(B^{f^1_{R^2}})$ and $b_j = \max \mathrm{Pr}_j(B^{f^1_{R^2}})$. We show that $\prod_{j=1}^m \{a_j, b_j\} \subseteq B^{f^1_{R^2}}$ by contradiction. Without loss of generality, suppose that $(b_1, \cdots, b_m) \notin B^{f^1_{R^2}}$. Pick $\tilde{\tilde{R}} \in \mathbb{S}$ which satisfies $p(\tilde{\tilde{R}}) = (b_1, \cdots, b_m)$. Let $h \in B^{f^1_{R^2}}$ satisfy $f^1(\tilde{\tilde{R}}, R^2) = h$. Since $h \neq (b_1, \cdots, b_m)$,

$$\exists j' \in M \text{ s.t. } h_{j'} < b_{j'}.$$

Since $b_{j'} = \max \mathrm{Pr}_j(B^{f^1_{R^2}})$, there exists $h' \in B^{f^1_{R^2}}$ such that $h'_{j'} = b_{j'}$. Hence, by Lemma 3,

$$\exists R^1 \in \mathbb{S} \text{ s.t. } p(R^1) = (b_1, \cdots, b_m) \wedge h' P(R^1) h$$

because $|b_{j'} - h'_{j'}| < |b_{j'} - h_{j'}|$. However this implies that

$$f^1(R^1, R^2) \neq h.$$

This is a contradiction because f satisfies weak peak-onliness. □

Proof of Theorem 2. (1) Note that f satisfies weak peak-onliness by Lemma 1 in Morimoto, Ching and Serizawa (2008) and Lemma 3 in Amorós (2002). We prove the conclusion by contradiction. Suppose that for some $g \in \Gamma_{\mathrm{SP}}$, g dom f but $f \neq g$. Then

$$(***) \quad \exists R^2 \in \mathbb{S} \text{ s.t. } B^{f^1_{R^2}} \subsetneq B^{g^1_{R^2}}.$$

If not, then $B^{f^1_{R^2}} = B^{g^1_{R^2}}$ for all $R^2 \in \mathbb{S}$ and $B^{f^1_{R^2}}$ is a direct product of closed interval by Lemma 6. This implies that $\#\tau(R^1, B^{f^1_{R^2}}) = 1$ for all $R^1 \in \mathbb{S}$ by single-peakedness. Then, we have $f^1(R^1, R^2) = g^1(R^1, R^2)$ for all $R^1 \in \mathbb{S}$ by Lemma 1. However this contradicts the assumption $g \neq f$. Hence $(***)$ holds.

We have $\tilde{x} \in \prod_{j=1}^m [0, \Omega_j]$ such that $\tilde{x} \in B^{g^1_{R^2}}$ and $\tilde{x} \notin B^{f^1_{R^2}}$. Let $\tilde{R} \in \mathbb{S}$ satisfy $p(\tilde{R}) = \tilde{x}$. By Lemma 6, for each $j \in M$, there exist $a_j, b_j \in [0, \Omega_j]$ such that $B^{f^i_R} = \prod_{j=1}^m [a_j, b_j]$. Then for each $j \in M$, one of the following three holds;

$$\begin{aligned}
&\text{(i)} \;\; \tilde{x}_j < a_j \;\; (\Leftrightarrow \Omega_j - a_j < \Omega_j - \tilde{x}_j),\\
&\text{(ii)} \;\; \tilde{x}_j \in [a_j, b_j] \;\; (\Leftrightarrow \Omega_j - \tilde{x}_j \in [\Omega_j - b_j, \Omega_j - a_j]),\\
&\text{(iii)} \;\; b_j < \tilde{x}_j \;\; (\Leftrightarrow \Omega_j - \tilde{x}_j < \Omega_j - b_j).
\end{aligned}$$

Let $y \in B^{f^1_{R^2}}$ be defined by the following; for each $j \in M$, $y_j = a_j$ if (i) holds, $y_j = \tilde{x}_j$ if (ii) holds and $y_j = b_j$ if (iii) holds. Obviously, $\tau(\tilde{R}, B^{f^1_{R^2}}) = \{y\}$. Hence,

$$f^1(\tilde{R}, R^2) = y \;\text{ and }\; g^1(\tilde{R}, R^2) = \tilde{x}.$$

Now let's consider the allotment for individual 2. Note that since f satisfies *respect for unanimity*, $\Omega - p(R^2) \in B^{f^1_{R^2}}$. Hence $p(R^2) \in sym(B^{f^1_{R^2}}) = \prod_{j=1}^m [\Omega_j - b_j, \Omega_j - a_j]$. By the definition of y and $\tilde{x}$, for each $j \in M$,

$$\begin{aligned} \text{(i)} &\;\Rightarrow\; \Omega_j - y_j = \Omega_j - a_j, \\ \text{(ii)} &\;\Rightarrow\; \Omega_j - y_j = \Omega_j - \tilde{x}_j, \\ \text{(iii)} &\;\Rightarrow\; \Omega_j - y_j = \Omega_j - b_j. \end{aligned}$$

Since $p_j(R^2) \in [\Omega_j - b_j, \Omega_j - a_j]$ for all $j \in M$,

$$\forall j \in M : p_j(R^2) \le \Omega_j - y_j \le \Omega_j - \tilde{x}_j \;\vee\; \Omega_j - \tilde{x}_j \le \Omega_j - y_j \le p_j(R^2).$$

Because $\tilde{x} \neq y$, $\Omega - \tilde{x} \neq \Omega - y$. By the single-peakedness of R^2, $(\Omega - y)P(R^2)(\Omega - \tilde{x})$. Feasibility condition implies that

$$f^2(\tilde{R}, R^2) = \Omega - y \;\text{ and }\; g^2(\tilde{R}, R^2) = \Omega - \tilde{x}.$$

However this contradicts that g dom f.

(2) Let f be a rule which satisfies *WP* and *symmetry*. We prove the conclusion by contradiction. Suppose that $\mathbf{R} = (R^1, R^2) \in \mathbb{S}^N$ satisfies $R^1 = R^2$ and $f(\mathbf{R}) \neq (\frac{\Omega}{2}, \frac{\Omega}{2})$. By *symmetry*, $f^1(\mathbf{R})I(R^1)f^2(\mathbf{R})$. Hence, by single-peakedness of R^1,

$$\neg[\forall j \in M : p_j(R^1) \le f^1_j(\mathbf{R}) \le f^2_j(\mathbf{R}) \vee f^2_j(\mathbf{R}) \le f^1_j(\mathbf{R}) \le p_j(R^1)].$$

By Lemma 4, there exist $\hat{R} \in \mathbb{S}$ such that $p(\hat{R}) = p(R^1)$ and $f^2(\mathbf{R})P(\hat{R}^1)f^1(\mathbf{R})$. Define $\hat{\mathbf{R}} = (\hat{R}, \hat{R})$. By *WP*, $f(\hat{\mathbf{R}}) = f(\mathbf{R})$. However this contradicts the *symmetry* of f.

(3) Obvious. □

Appendix C : SP, ER and WP rules

In this appendix, I show that there are many rules which satisfies *SP*, *ER* and *WP* except for the equal division rule and the uniform rule.

Let $m = 1$ and $\epsilon \in [0, \frac{\Omega}{2}]$. $f^{(\epsilon)}$ is the rule which satisfies the following; for each $R^2 \in \mathbb{S}$,

$$B^{f^{(\epsilon)1}_{R^2}} = \begin{cases} [\frac{\Omega}{2} - \min\{\epsilon, p(R^2) - \frac{\Omega}{2}\}, \frac{\Omega}{2}] & \text{if } \frac{\Omega}{2} \le p(R^2), \\ [\frac{\Omega}{2}, \frac{\Omega}{2} + \min\{\epsilon, \frac{\Omega}{2} - p(R^2)\}] & \text{if } p(R^2) \le \frac{\Omega}{2}, \end{cases}$$

and $f^{(\epsilon)1}(R^1, R^2) \in \tau(R^1, B^{f^{(\epsilon)1}_{R^2}})$. Then, obviously for each $R \in \mathbb{S}$, $B^{f^{(\epsilon)2}_R} = B^{f^{(\epsilon)1}_R}$.

Now I introduce *SP*, *ER* and *WP* rules with $m \ge 2$. Let $(\epsilon_1, \cdots, \epsilon_m) \in \prod_{j=1}^m [0, \frac{\Omega_j}{2}]$. $f^{(\epsilon_1, \cdots, \epsilon_m)}$ is the rule allotting each commodities by the rule above. Then $f^{(\epsilon_1, \cdots, \epsilon_m)}$ satisfies *SP*, *ER* and *WP*. [17]

[17] Note that if $(\epsilon_1, \cdots, \epsilon_m) = (0, \cdots, 0)$, then $f^{(\epsilon_1, \cdots, \epsilon_m)}$ is the equal division rule, and if $(\epsilon_1, \cdots, \epsilon_m) = (\frac{\Omega_1}{2}, \cdots, \frac{\Omega_m}{2})$, then $f^{(\epsilon_1, \cdots, \epsilon_m)}$ is the uniform rule.

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