Convenience Yield Revisted: Using production based discrete-time model

Katsushi Nakajima* Faculty of Commerce, Waseda University
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Abstract

This paper is to analyze commodity price and also reconsider what convenience yield is. Our model in this paper considers two types of players that is a firm, which use an input commodity to produce an output commodity and trade futures to hedge, and speculators which consumes the output commodity while earning profits by trading commodities. This model imply that the convenience yield can be decomposed in two components which are the “yield” part and the “cost” part. We also found that convenience yields can be interpreted in at least three ways from the relations between spot commodity price, future spot price, and futures price. The optimal amount of spot commodities and futures are derived. Introducing the speculator, we can see that convenience yields are implicitly included in the intertemporal adjustment term.

1 Introduction

Commodity such as energy, metal, and agricultural products, are still one of the main issue in economics and also in politics. Trade policy on agricultural and energy policy influence commodity prices and thus people’s lifestyles.

However, commodity price and its model are not yet fully explored. The main factor which determines commodity price is the convenience yield. Convenience yield, which was first indicated by Kaldor (1939) and explored by

*Faculty of Commerce, Waseda University, 1-6-1 Nishi-waseda, Shinjuku-ku, Tokyo, 169-8050, Japan. E-mail: knakajima@aoni.waseda.jp.
Working (1949), is a very abstract concept.\footnote{In their paper, convenience yield is separated from storage cost. However, in this paper we include storage cost in the convenience yield. Therefore, we will be analyzing the “net convenience yield.”} It is an yield which the owner of the spot commodity earns compared to the owner of the right of commodity such as futures.

Working (1949) pointed out that convenience yield increases as the storage decreases or in other words the market tightens. However, Kaldor (1939) and Working (1949) analyzed convenience yield using primitive tools no more than expectation. Gibson and Schwartz (1990) proposed a benchmark model for commodity price which includes mean-reverting convenience yield. In their paper, they used no-arbitrage argument and derived a partial differential equation for derivative on commodity such as futures. They also empirically analyzed convenience yield for crude oil. An elementary introduction of convenience yield exist is provided by Hull (2009). He show that convenience yield exist because there are no demand of short selling of commodity which is different from investment assets.

This paper is to analyze commodity price and also reconsider what convenience yield is. Our model in this paper considers two types of players that is a firm, which use an input commodity to produce an output commodity and trade futures to hedge, and speculators which consumes the output commodity while earning profits by trading commodities. The firm can also store input commodity by paying storage cost. The model is a discrete-time dynamic programming model. We emphasize that the model is based on a firm which use and produce a commodity. These kind of model which include supply and demand of physical commodity are not yet fully investigated. However, there are some papers which consider economic model. Kanamura (2009) which provide a volatility model using supply and demand curves. Casassus et al. (2013) study commodity prices including production rates and utility functions. They have found that convenience yields are the marginal productivity rates. Our model is different from these models which define two types of players explicitly which should add some incite to convenience yield and provide another view.

In Section 2, we set up the model. In Section 3, we discuss the result of the model. We provide some equations relating spot commodity price, future spot commodity price, and futures price. We also derive the optimal amount of buying and using commodity and the optimal hedging amount of using futures. Moreover, if we introduce speculators, we can see that convenience yield is implied in the intertemporal factor. Section 4 concludes.

2 The Model

2.1 A Firm

We consider a firm which use commodity 1, e.g. coal, to produce commodity 2, e.g. electricity. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,...,T}, P)\) be a filtered probability space. Let \(Q\)
be continuously differentiable production function for commodity 1 which use commodity 2. We assume that $Q$ is a concave, nondecreasing function, and its derivative function is strictly decreasing. The prices of spot commodities at time $t$ are $S_i(t), i = 1, 2$. These processes are positive, $\{F_t\}$-adapted. Commodity 2 can be stored and there are futures contract for commodity 2 which matures at each time $t$. The prices of futures commodities at time $t$ which mature at $u$ are $F_i(t, u), i = 1, ..., N, t = 0, ..., T, u = t + 1, ..., T$, positive, $\{F_t\}$-adapted for each $u$. The futures commodity is physically delivered with spot commodity at maturity. Therefore, $F(t, t) = S(t)$. Let $R$ be continuously differentiable cost function of storage of physical commodity 2. We assume that $R$ is a strictly convex, nonincreasing function, and its derivative function is strictly increasing. The interest rate is $r$ which is fixed. Therefore the price of the bank account at time $t$ is $S_0(t) = \exp(rt)$. We denote $E_t(\cdot)$ as the conditional expectation given $F_t$.

We use the following notations. $q_{S,1}(t)$ is the amount of spot commodity 2 bought at time $t$, $q_{S,2}(t)$ is the amount of spot commodity 2 used at time $t$, $q_{F,1}(t, s), s = 0, ..., T$ is the amount of future commodity 2 which matures at time $s$ bought at time $t$. Note that the amount of purchased future commodity 2 $q_{F,1}(t, s), s = 0, ..., t$ which matures before time $t$ are 0, because there are no futures traded after the time matures. The amount of future commodity 2 can be used as spot commodity after it matures. We use the notation $M = 2 + 2(T + 1)$-dimensional variable $q(t) = (q_{S,1}(t), q_{S,2}(t), (q_{F,1}(t, s))_{s=0, ..., T})$ for these amounts and $q = (q(t))_{t=0, ..., T}$. The amount of storage of spot commodity is $x_S(t) = x_S(t - 1) + q_{S,1}(t - 1) - q_{S,2}(t - 1)$ with $x_S(0) = 0$. The amount of storage of futures commodity which matures at time $s$ are $x_F(t, s) = x_F(t - 1, s) + q_{F,1}(t - 1, s), s = t, ..., T$ with $x_F(0, s) = 0, s = 0, ..., T$. The amount of storage of futures commodity which reached maturity $x_F(t, s), s = 0, ..., t$ are spot commodities and the firm can use it for production. The amount of storage of futures commodity do not change $x_F(t, s) = x_F(t, t), s = 0, ..., t - 1$ after it matures because there are no futures to buy. We use the notation $x(t) = (x_S(t), (x_F(t, s))_{s=0, ..., t})$ for storage.

A firm’s objective is to maximize its profit.

$$
\max_{q \in Q} \quad E_0 \left[ \sum_{t=0}^{T} e^{-rt} \left( Q(q_{S,2}(t))S_1(t) - q_{S,1}(t)S_2(t) \\
- R\left( x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t} x_F(t, s) \right) \right) \right. \\
- \left. \sum_{t=0}^{T} \sum_{s=t+1}^{T} e^{-rt} q_{F,1}(t, s) F(t, s) \right] \\
+ e^{-rT} \left( x_S(T) + q_{S,1}(T) - q_{S,2}(T) + \sum_{s=0}^{T} x_F(T, s) \right) S_2(T). \quad (1)
$$
where

\[
Q = \{ q : q \text{ is } \mathcal{F}_t \text{-adapted process} \}
\]

\[
x_S(t) + q_S,1(t) - q_S,2(t) + \sum_{s=0}^{t} x_F(t, s) \geq 0, t = 0, \ldots, T,
\]

\[
q_S,2(t) \geq 0, t = 0, \ldots, T,
\]

\[
x_F(t, s) + q_F,1(t, s) \geq 0, t = 0, \ldots, T, s = t + 1, \ldots, T\}
\]

The first row is the firm’s profit from its main business. The second row is the storage cost of spot commodity 2. The third row is the cost of purchasing futures contract on commodity 2. The last term is the income from dispose of storage.

Note that the firm do not short sell spot commodities or futures contract. This is why convenience yield exist. If firm short sell spot commodities and futures contract, there will be no convenience yield. Also, it is assumed that futures can be stored but does not need storage cost if it is not matured.

We also need the following assumption to solve the optimal amounts.

**Assumption 1.**

1. There exist a function \( g_R \in L^1(Ω, P) \) where \(|R| \leq g_R\).
2. There exist a function \( h_R \in L^1(Ω, P) \forall x \in R^{2+T-t}, t = 0, \ldots, T \) where \(|R'| \leq h_R\).

See Appendix for the necessary conditions for problem (1) and its optimal solution.

### 2.2 Speculators’ utility maximization problem

Now we consider \( I \) individuals. Each individual consume commodity 1 and speculate by trading commodities and futures contract. Her utility function is \( u_i \) which is bounded, continuously differentiable, strictly increasing, and strictly concave. She owns a share of the firm which the share is fixed. Therefore a part of the firm’s profit \( \theta_i, x(t) \) at time \( t \) will be her income. Her endowment is \( e_i(t) = (e_i,0(t), e_i,1(t), e_i,2(t)) \) at time \( t \) which is the amount corresponding to bank account, commodity 1, and commodity 2.

Let \( c_i(t) \) be the amount of consumption of commodity 1 at time \( t \) Let \( C \) be the space of nonnegative adapted processes in \( R \) for consumption and \( Θ \) be the space of adapted processes in \( R^4 \) for trading strategies. Let

\[
S(t) = (S_0(t), S_1(t), S_2(t)),
\]

\[
θ_i,S(t) = (θ_i,S_0(t), θ_i,S_1(t), θ_i,S_2(t)),
\]

\[
θ_i = (θ_i,S(t), θ_i,F(t, s)_{s=0,...,T})
\]

\(^2L^1(Ω, \mathcal{F}, P) \) is a space of integrable function on \( Ω \) with respect to the measure \( P \).
The individual maximize the following expected utility.

$$\max_{(c_{i,1}, \theta_i) \in \mathcal{A}} E_0 \left[ \sum_{t=0}^{T} e^{-r t} u_i(t, c_{i,1}(t)) \right]$$

(2)

where

$$\mathcal{A} = \left\{ (c_{i,1}, \theta_i) \in C \times \Theta : c_{i,1}(t) S_1(t) = \theta_i x(t) + S(t) \cdot \epsilon_i(t) + w_i(t) - \theta_i S(t) \cdot S(t) - \sum_{s=t+1}^{T} \theta_i, x(t, s) F(t, s) \right\}.$$ 

$$w_i(0) = w_i,$$

$$w_i(t) = \theta_i S(t-1) \cdot S(t) + \sum_{s=t}^{T} \theta_i, x(t-1, s) F(t, s), t = 1, ..., T.$$

We assume the following.

**Assumption 2.**

1. There exist a function $g_u \in L^1(\Omega, P)$ where $|u| \leq g_u$.

2. There exist a function $h_u \in L^1(\Omega, P)|u| \in R^{2n+T-t}$, $t = 0, ..., T$ where $|\partial u(t, \cdot)| \leq h_u$, $t=0, ..., T$.

These assumptions are needed to solve the problem (2). We discuss the results in the next section.

**3 Discussion**

Our model imply some important results. From equations (10), (11), (12) in the appendix, we can see that spot prices and futures prices are related as follows.

$$F(t, s) = E_t[e^{-r(T-s)} S_2(T)] - E_t \left[ \sum_{v=s}^{T} e^{-r(v-s)} R \right] + e^{r t} \lambda_{F,1}(t, s, x(t))$$

$$s = t + 1, ..., T$$

$$S_2(t) = e^{-r(T-t)} E_t[S_2(T)] - E_t \left[ \sum_{v=t}^{T} e^{-r(v-t)} R \right] + e^{r t} \lambda_{S,1}(t, x(t))$$

$$S_2(t) = e^{-r(s-t)} F(t, s) - E_t \left[ \sum_{v=t}^{s-1} e^{-r(v-t)} R \right] + e^{r t} \lambda_{S,1}(t, x(t)) - e^{r t} \lambda_{F,1}(t, s, x(t))$$

$$s = t + 1, ..., T.$$
These convenience yields can be decomposed in two components which are the “cost” part of the storage and the “yield” part of the storage. The cost part of these convenience yields are the marginal cost of storage $R'$. The yield part are the Lagrange multipliers $\lambda_{S,1}$ and $\lambda_{F,1}$. Recall that $\lambda_{S,1}$ and $\lambda_{F,1}$ are nonnegative. The third equation which the yield part is $e^{-t} (\lambda_{S,1}(t, x(t)) - \lambda_{F,1}(t, s, x(t)))$.

These equations imply that the convenience yields can be interpreted in at least three ways. The first is the convenience yield on future spot prices compared to futures price. This occurs by the constraints on short sale of futures. Therefore, if the firm do not have any constraint on short sale of futures then this convenience yield will not exist.

The second is the convenience yield on spot prices compared to future spot prices. This occurs by the constraints on short sale of spot commodity. Thus, this convenience yield is the fundamental yield.

And the last is the convenience yield on spot prices compared to futures prices. It is intuitive that the convenience yield on spot prices compared to futures prices can be derived by the other two convenience yields.

Note that the Lagrange multiplier is the shadow price.

$$\lambda_{S,1}(t, x(t)) = \frac{\partial p(t, x(t), 0)}{\partial S_{1}},$$

$$\lambda_{F,1}(t, s, x(t)) = \frac{\partial p(t, x(t), 0)}{\partial F_{1, s}}, s = t + 1, ..., T,$$

where

$$p(t, x(t), l) = \sup \{ W(t, x(t), q(t)) : q(t) \in \mathbb{R}^{3+T},$$

$$x_s(t)+q_{S,1}(t)-q_{S,2}(t)+\sum_{s=0}^{t} x_F(t, s) \geq l_{S,1},$$

$$q_{S,2}(t) \geq l_{S,2},$$

$$x_F(t, s) + q_{F,1}(t, s) \geq l_{F,1,s}, s = t + 1, ..., T\},$$

$$l = (l_{S,1}, l_{S,2}, \{l_{F,1,s}\}_{s=t+1, ..., T})$$

This implies that the yield part of convenience yield is determined by the profit stemmed on the availability of spot or futures. In order to analyze the relation between the convenience yields and the storage amounts, we need to analyze $\lambda_{S,1}$ and $\lambda_{F,1}$. 
We also derived the optimal amount of spot commodities and futures.

\[ q_{S,1}^*(t) = I_{V_{S,1}}(-e^{-rt}S_2(t) + \lambda_{V_{S,1}}(t)) - xS(t) + q_{S,2}^*(t) - \sum_{s=0}^{t} x_F(t, s) \]

\[ q_{S,2}^*(t) = I_Q \left( -S_2(t) - \lambda_{S,2}(t, x(t)) \right) S_1(t) \]

\[ q_{F,1}(t, s) = I_{V_{F,1}}(-e^{-rs}F(t, s) + \lambda_{V_{F,1}}(t, s)) - xS(t) + q_{S,2}^*(t) - \sum_{s=0}^{t} x_F(t, s), s = t + 1, ..., T. \]

The firm buys \( q_{S,1}^*(t) \) and use \( q_{S,2}^*(t) \) commodity 2, and trades futures \( q_{F,1}(t, s) \). \( q_{F,1}(t, s) \) is the hedging strategy for the firm. Although the optimal amount used \( q_{S,2}^*(t) \) is determined by the two commodity prices, the optimal amount of buying \( q_{S,1}^*(t) \) do not depend on \( S_1(t) \) explicitly.

We now turn to the result on the speculator. From (18) and (19) in the appendix, we have

\[ S_i(t) = E_t \left[ \frac{\partial u(t+1, c_i^*(t+1))}{\partial c_1} S_i(t+1) \right] \frac{S_i(t+1)}{S_i(t)} \] (3)

\[ F(t, s) = E_t \left[ \frac{\partial u(t+1, c_i^*(t+1))}{\partial c_1} S_i(t+1) \right] \frac{S_i(t+1)}{S_i(t)} \] (4)

or

\[ S_i(t) = E_t \left[ \frac{\partial u(T, c_i(T))}{\partial c_i} S_i(T) \right] \frac{S_i(T)}{S_i(t)} \] (5)

\[ F(t, s) = E_t \left[ \frac{\partial u(T, c_i(T))}{\partial c_i} S_i(T) \right] \frac{S_i(T)}{S_i(t)} \] (6)

Note that \( S_1(t) \) is used as the numeraire price. This result is not surprising which is just a modification of ordinary intertemporal price relation. We can see that it seems that convenience yield do not exist explicitly among the prices. However, comparing to (11) and (10), we may interpret that it is included
implicitly. Concretely speaking we have

\[
E_t \left[ \left( \frac{\partial u(T,c_1^*(T))}{\partial c_1} \frac{1}{S_1(t)} - e^{-r(T-t)} \right) S_2(T) \right]
\]

\[
= -E_t \left[ \sum_{v=t}^{T} e^{-r(v-t)} R_v \right] + e^{rt} \lambda_{s,1}(t, x(t))
\]

\[
E_t \left[ \left( \frac{\partial u(T,c_1^*(T))}{\partial c_1} \frac{1}{S_1(t)} - e^{-r(T-s)} \right) F(T, s) \right]
\]

\[
= -E_t \left[ \sum_{v=s}^{T} e^{-r(v-s)} R_v \right] + e^{rs} \lambda_{s,1}(t, s, x(t)), s = t + 1, ..., T.
\]

Therefore, the convenience yields are implicitly included in the intertemporal adjustment term

\[
\frac{\partial u(T,c_1^*(t))}{\partial c_1} \frac{1}{S_1(t)}.
\]

4 Conclusion

In this paper, we have modeled a firm which use input commodity to produce output commodity and also trade futures on input commodity. The firm can also store input commodity by paying storage cost. This model imply that the convenience yield can be decomposed in two components which are the “yield” part and the “cost” part. We also found that convenience yields can be interpreted in at least three ways from the relations between spot commodity price, future spot price, and futures price. The optimal amount of spot commodities and futures are derived. Introducing the speculator, we can see that convenience yields are implicitly included in the intertemporal adjustment term.

There are many additional analysis needed for future research. We list a few below. First, it is interesting to analyze the relation of convenience yield and storage amount. Second, the model can be enhanced to include multiple commodities for inputs and outputs. Third, the model can be enhanced to continuous-time model and we can compare with other model such as Gibson and Schwartz (1990).
5 Appendix

5.1 Characterization of the Convenience Yield

In this section, we derive the necessary condition for problem (1). The Bellman’s equation\(^3\) for this problem at time \(t\) is

\[
V_p(t, x(t)) = \max_{q(t) \in \mathcal{Q}(t, x(t))} e^{-rT}(Q(q_{S,2}(t))S_1(t) - q_{S,1}(t)S_2(t) \\
- R\left(x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t} x_F(t, s)\right) \\
- \sum_{s=t+1}^{T} e^{-rT}q_{F,1}(t, s)F(t, s) + E_t[V_p(t + 1, x(t + 1))],
\]

(7)

where

\[
Q(t, x(t)) = \{ q(t) = (q_{S,1}(t), q_{S,2}(t), \{q_{F,1}(t, s)\}_{s=t+1, \ldots, T}) : \\
x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t} x_F(t, s) \geq 0, \\
q_{S,2}(t) \geq 0, \\
x_F(t, s) + q_{F,1}(t, s) \geq 0, s = t + 1, \ldots, T, \}
\]

and

\[
V_p(T, x(T)) = \max_{q(T) \in \mathcal{Q}(T, x(T))} e^{-rT}\left(Q(q_{S,2}(T))S_1(T) - q_{S,1}(T)S_2(T) \\
- R\left(x_S(T) + q_{S,1}(T) - q_{S,2}(T) + \sum_{s=0}^{T} x_F(T, s)\right) \\
+ e^{-rT}\left(x_S(T) + q_{S,1}(T) - q_{S,2}(T) + \sum_{s=0}^{T} x_F(T, s)\right)S_2(T)\right)
\]

(8)

We can rewrite the problem at time \(T\) as

\[
V_p(T, x(T)) = \max_{q(T) \in \mathcal{Q}(T, x(T))} W_p(T, x(T), q(T))
\]

\(^3\)For Bellman’s equation, see Stokey and Lucas (1989), Section 9.1, or Bertsekas (2005), Section 1.3.
The Karush-Kuhn-Tucker conditions\(^4\) at time \(T\) are

\[
x_S(T) + q_{S,1}^*(T) - q_{S,2}^*(T) + \sum_{s=0}^{T} x_F(T, s) \geq 0,
\]

\[
q_{S,2}^*(T) \geq 0,
\]

\[
\lambda_{S,1}(T, x(T)) \geq 0,
\]

\[
\lambda_{S,1}(T, x(T))(x_S(T) + q_{S,1}^*(T) - q_{S,2}^*(T)) = 0,
\]

\[
\lambda_{S,1}(T, x(T))q_{S,2}^*(T) = 0,
\]

\[
-S_2(T) - R\left( x_1(T) + q_{S,1}^*(T) - q_{S,2}^*(T) + \sum_{s=0}^{T} x_F(T, s) \right)
\]

\[
+e^{-rT}\left[ R[T] + \lambda_{S,1}(T, x(T)) \right] = 0
\]

\[
Q'(q_{S,2}^*(T))S_1(T) + R\left( x_1(T) + q_{S,1}^*(T) - q_{S,2}^*(T) + \sum_{s=0}^{T} x_F(T, s) \right)
\]

\[
-e^{-rT}\left[ R[T] - \lambda_{S,1}(T, x(T)) + \lambda_{S,2}(T, x(T)) \right] = 0
\]

Moreover, the Bellman equation at time \(t\) is

\[
V_p(t, x(t)) = \max_{q(t) \in Q(t, x(t))} W_p(t, x(t), q(t))
\]

\(^4\)See Rockafellar (1970), Section 28, Theorem 28.3 or Boyd and Vandenberghe (2004), Section 5.3.3.
The Karush-Kuhn-Tucker conditions at time $t$ are

$$W_p(t, x(t), q(t)) = e^{-rt} \left( Q(q_{S,2}(t))S_1(t) - q_{S,1}(t)S_2(t) - \sum_{s=t+1}^{T} e^{-rs} q_{F,1}(t, s)F(t, s) \right)$$

$$\begin{align*}
- R \left( x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t} x_f(t, s) \right) \\
+ E_t \left[ \sum_{v=t+1}^{T} e^{-rv} \left( Q(q_{S,2}^*(v))S_1(v) - q_{S,1}^*(v)S_2(v) \right) \right] \\
- R \left( x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{w=t+1}^{v} q_{S,1}^*(w) - q_{S,2}^*(w) \right) \\
+ \sum_{s=0}^{t} x_f(t, s) + q_{F,1}(t, s) + \sum_{v=t+1}^{T} q_{F,1}^*(v, w, s) \right) \\
- \sum_{v=t+1}^{T} \sum_{s=0}^{t} e^{-rs} q_{F,1}^*(v, s)F(v, s) \\
+ e^{-rt} \left( x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{v=t+1}^{T} q_{S,1}^*(v) - q_{S,2}^*(v) \right) \\
+ \sum_{s=0}^{t} x_f(t, s) + q_{F,1}(t, s) + \sum_{v=t+1}^{T} q_{F,1}^*(v, s) \right) S_2(T) \right].
\end{align*}$$

The Karush-Kuhn-Tucker conditions at time $t$ are

$$x_S(t) + q_{S,1}^*(t) - q_{S,2}^*(t) + \sum_{s=0}^{t} x_f(t, s) \geq 0,$$

$$q_{S,2}^*(t) \geq 0,$$

$$x_f(t, s) + q_{F,1}^*(t, s) \geq 0, \quad s = t + 1, \ldots, T,$$

$$\lambda_{S,1}(t, x(t)) \geq 0, \quad \lambda_{F,1}(t, s, x(t)) \geq 0, \quad s = t + 1, \ldots, T,$$

$$\lambda_{S,1}(t, x(t))(x_S(t) + q_{S,1}(t) - q_{S,2}(t)) = 0, \quad \lambda_{S,2}(t, x(t))q_{S,2}^*(t) = 0,$$

$$\lambda_{F,1}(t, s, x(t))(x_f(t, s) + q_{F,1}(t, s)) = 0, \quad s = t + 1, \ldots, T,$$

$$-e^{-rt}S_2(t) + E_t[e^{-rT}S_2(T)] - E_t \left[ \sum_{v=t}^{T} e^{-rv} R' \right] + \lambda_{S,1}(t, x(t)) = 0,$$

$$e^{-rt}Q'(q_{S,2}^*(t))S_1(t) - E_t[e^{-rT}S_2(T)] + E_t \left[ \sum_{v=t}^{T} e^{-rv} R' \right] - \lambda_{S,1}(t, x(t)) + \lambda_{S,2}(t, x(t)) = 0,$$

$$-e^{-rs}F(t, s) + E_t[e^{-rT}S_2(T)] - E_t \left[ \sum_{v=s}^{T} e^{-rv} R' \right] + \lambda_{F,1}(t, s, x(t)) = 0, \quad s = t + 1, \ldots, T.$$
Suppose that Assumption 1 holds. Suppose that there exist an
For strictly concavity, see Stokey and Lucas (1989), Theorem 9.8 and
5.2 Solving the profit maximization problem

We now solve the optimal production and trade amount.

We need the following lemma.

Lemma 5.1. Suppose that Assumption 1 holds. Suppose that there exist an
optimal solution $q^*$ which is in $\text{int} Q$. For any $t$, $V_p(t, \cdot)$ is a strictly concave and
continuously differentiable function.

\[
\frac{\partial V_p(t, x(t))}{\partial x_i} = -R' \left( x_S(t) + q^*_S(t) - q^*_S(t) + \sum_{s=0}^{t} x_F(t, s) \right) + E_t \left[ \frac{\partial V_p(t+1, x^*(t+1))}{\partial x_i} \right] , t = 0, \ldots, T - 1
\]

\[
\frac{\partial V_p(T, x(T))}{\partial x_i} = -R' \left( x_S(T) + q^*_S(T) - q^*_S(T) + \sum_{s=0}^{T} x_F(T, s) \right) + e^{-RT} S_2(T)
\]

where $x^*(t + 1)$ is the storage amount associated with $q^*(t)$.

Proof. For strictly concavity, see Stokey and Lucas (1989), Theorem 9.8 and
Exercise 9.7.d.

For the differentiation, we prove it by mathematical induction. Let us consider $T$. Since $R$ is continuously differentiable, $V_p(T, x(T))$ is continuously differentiable.

\[
\frac{\partial V_p(T, x(T))}{\partial x_i} = -R' \left( x_S(T) + q^*_S(T) - q^*_S(T) + \sum_{s=0}^{T} x_F(T, s) \right) + e^{-RT} S_2(T)
\]

Now consider $t$. Suppose that $V_p(t + 1, x(t + 1))$ is continuously differentiable. By mathematical induction, we can interchange the derivative and expectation.
operator using the Assumption 1.
\[
\frac{\partial E_i[V_p(t+1,x(t+1))]}{\partial x_i} = E_i \left[ \frac{\partial V_p(t+1,x(t+1))}{\partial x_i} \right]
\]
Therefore
\[
\frac{\partial V_p(t,x(t))}{\partial x_i(t)} = -R' \left( x_S(t) + q_{S,1}^*(t) - q_{S,2}^*(t) + \sum_{s=0}^{t} x_F(t,s) \right)
+ E_i \left[ \frac{\partial V_p(t+1,x^*(t+1))}{\partial x_i(t)} \right].
\]

The Karush-Kuhn-Tucker conditions for problem (7) is
\[
x_S(t) + q_{S,1}^*(t) - q_{S,2}^*(t) + \sum_{s=0}^{t} x_F(t,s) \geq 0,
q_{S,2}(t) \geq 0, x_F(t,s) + q_{F,1}^*(t,s) \geq 0, s = t + 1, ..., T,
\lambda_{V_p,S,1}(t) \geq 0, \lambda_{V_p,S,2}(t) \geq 0, \lambda_{V_p,F,1}(t,s) \geq 0, s = t + 1, ..., T,
\lambda_{V_p,S,1}(t) \left( x_S(t) + q_{S,1}^*(t) - q_{S,2}^*(t) + \sum_{s=0}^{t} x_F(t,s) \right) = 0,
\lambda_{V_p,S,2}(t) q_{S,2}(t) = 0, \lambda_{V_p,F,1}(t,s)(x_F(t,s) + q_{F,1}^*(t,s)) = 0, s = t + 1, ..., T,
-e^{-rt} S_2(t) - e^{-rt} R' + E_i \left[ \frac{\partial V_p(t+1,x^*(t+1))}{\partial q_{S,1}} \right] + \lambda_{V_p,S,1}(t) = 0,
-e^{-rt} Q' S_1(t) + e^{-rt} R' + E_i \left[ \frac{\partial V_p(t+1,x^*(t+1))}{\partial q_{S,2}} \right] - \lambda_{V_p,S,1}(t) + \lambda_{V_p,S,2}(t) = 0,
-e^{-rs} F(t,s) + E_i \left[ \frac{\partial V_p(t+1,x^*(t+1))}{\partial q_{F,1}} \right] + \lambda_{V_p,F,1}(t,s) = 0, s = t + 1, ..., T.
\]
where we denoted \( R' = R'(x_S(t) + q_{S,1}^*(t) - q_{S,2}^*(t) + \sum_{s=0}^{t} x_F(t,s)), t = 0, ..., T. \)
For the last three equations, we interchange the derivative and expectation operator by using the assumption on \( R'. \)

**Lemma 5.2.** There exist inverse functions to
\[
\phi_{S,1}(q_{S,1}(t)) = e^{-rt} R' \left( x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t} x_F(t,s) \right)
- E_i \left[ \frac{\partial V_p(t+1,x(t+1))}{\partial x_{S,1}} \right],
\phi_{F,1}(q_{F,1}(t,s)) = -E_i \left[ \frac{\partial V_p(t+1,x(t+1))}{\partial x_{F,1,s}} \right].
\]
where we define the inverse functions $I_{V_p,S;1}$ and $I_{V_p,F;1}$, respectively.

**Proof.** Since $-V_p(t+1,\cdot)$ is continuous convex function, then so is $-E_{t+1}[V_p(t+1,\cdot)]$. Combining this with $e^{-rt}R$ which is continuous convex function, we have an inverse function for

$$
\phi_{S,1}(q_{S,1}(t)) = e^{-rt}R\left(x_S(t) + q_{S,1}(t) - q_{S,2}(t) + \sum_{s=0}^{t}x_F(t,s)\right)
$$

from Theorem 23.5 and Corollary 23.5.1 of Rockafellar (1970). The other function is more easier to prove.

Since $Q$ is a continuous concave function, there exist an inverse function for $Q'$ from Theorem 23.5 and Corollary 23.5.1 of Rockafellar (1970) which satisfy $I_Q(-Q'(q_{S,2}(t))) = q_{S,2}(t)$. Thus,

$$
q_{S,2}(t) = I_Q\left(-\frac{S_2(t) - \lambda_{S_2}(t, x(t))}{S_1(t)}\right).
$$

From Lemma 5.2 we have

$$
q^*_{S,1}(t) = I_{V_p,S;1}(-e^{-rt}S_2(t) + \lambda_{V_p,S;1}(t)) - x_S(t) + q^*_{S,2}(t)
$$

$$
q^*_{F,1}(t,s) = I_{V_p,F;1}(-e^{-rs}F(t,s) + \lambda_{V_p,F;1}(t,s)) - x_S(t) + q^*_{S,2}(t)
$$

which is the optimal amount.

**5.3 Solving the maximization of utility**

In this section, we derive the necessary condition for problem (2). Here we omit the index $i$ since the argument is basically same for each individuals.

The Bellman’s equation for this problem at time $T$ is

$$
V_c(T, w(T)) = \max_{(c_1(T), \theta(T)) \in A(T, w(T))} u(T, c_1(T))
$$

where

$$
A(T, w(T)) = \{(c, \theta) \in \mathbb{R}_+ \times \mathbb{R}^{2+T-t} : c_1(T)S_1(T) = \pi(T) + S(T) \cdot c(T) + w(T) - \theta S(T) \cdot S(T)\}.
$$
The Karush-Kuhn-Tucker conditions are
\[ c_1^*(T) \geq 0, \lambda_{c,1}(T) \geq 0, \lambda_{c,1}(T)c_1^*(T) = 0 \]
\[ c_1^*(T)S_1(T) = \pi(T) + S(T) \cdot \epsilon(T) + w(T) - \theta_2^*(T) \cdot S(T) \]
\[ \frac{\partial u(T, c_1^*(T))}{\partial c_2} + \lambda_{c,1}(T) - \lambda_{c,2}(T)S_1(T) = 0. \]

The Bellman’s equation for this problem at time \( t \) is
\[
V_c(t, w(t)) = \max_{(c_1(t), \theta(t)) \in A(t, w(t))} u(t, c_1(t)) + \mathbb{E}_t \left[ V_c \left( t+1, \theta_S(t) \cdot S(t+1) + \sum_{s=t+1}^{T} \theta_F(t,s)F(t+1, s) \right) \right]
\]
where
\[ A(t, w(t)) = \{(c, \theta) \in \mathbb{R}_+ \times \mathbb{R}^{2+T-t} : c_1(t)S_1(t) = \pi(t) + S(t) \cdot \epsilon(t) + w(t) \]
\[ -\theta_S(t) \cdot S(t) - \sum_{s=t+1}^{T} \theta_F(t,s)F(t,s) \}. \]

**Lemma 5.3.** For any \( t \), \( V_c(t, \cdot) \) is strictly concave. If \((c_1^*, \theta^*)\) solves (16), and \( c_1^* \) is strictly positive, then \( V_c(t, \cdot) \) is continuously differentiable at \( w \) with derivative
\[
\frac{\partial V_c(t,w)}{\partial w} = \frac{\partial u(t, c_1^*)}{\partial c_1} \frac{1}{S_1(t)}
\]

**Proof.** For strictly concavity, see Stokey and Lucas (1989), Theorem 9.8 and Exercise 9.7.d.

The Bellman equation can be written as
\[
V_c(t, w(t)) = \max_{\theta(t) \in \Theta(t, w(t))} u(t, \left( \pi(t) + S(t) \cdot \epsilon(t) + w(t) - \theta_S(t) \cdot S(t) - \sum_{s=t+1}^{T} \theta_F(t,s)F(t,s) \right) / S_1(t) )
\]
\[+ \mathbb{E}_t [V_c(t+1, w^*(t+1))] \]
where
\[ \Theta(t, w(t)) = \{ \theta \in \mathbb{R}^{2+T-t} : S(t) \cdot \epsilon(t) + w(t) - \theta_S(t) \cdot S(t) \]
\[ - \sum_{s=t+1}^{T} \theta_F(t,s)F(t,s) \geq 0 \} \]

From Proposition 9.10 in Stokey and Lucas (1989),
\[
\frac{\partial V_c(t,w)}{\partial w} = \frac{\partial u(t, \pi(t) + S(t) \cdot \epsilon(t) + w(t) - \theta_S(t) \cdot S(t) - \sum_{s=t+1}^{T} \theta_F(t,s)F(t,s))}{\partial w} \frac{1}{S_1(t)}
\]
\[= \frac{\partial u(t, c_1^*)}{\partial c_2} \frac{1}{S_1(t)}. \]
The Karush-Kuhn-Tucker conditions at time $t$ are

$$c_1(t) \geq 0, \lambda_{c, 1}(t) \geq 0, \lambda_{c, 1}(t) c_1^*(t) = 0$$

$$c_1^*(t) S_1(t) = \pi(t) + S(t) \cdot c(t) + w(t) - \theta_{x_2}^*(t) \cdot S(t) - \sum_{s=t+1}^T \theta_P^*(t, s) F(t, s)$$

$$\frac{\partial u(t, c_1^*(t))}{\partial c_2} + \lambda_{c, 1}(t) - \lambda_{c, 2}(t) S_1(t) = 0$$

$$E_t \left[ \frac{\partial V_c(t + 1, w^*(t + 1))}{\partial w} S_1(t + 1) \right] - \lambda_{c, 2}(t) S_1(t) = 0$$

$$E_t \left[ \frac{\partial V_c(t + 1, w^*(t + 1))}{\partial w} F(t + 1, s) \right] - \lambda_{c, 2}(t) F(t, s) = 0.$$

where we interchanged the partial differential and expectation operator using Assumption 2.

Therefore, if $c_1^*(t) > 0$, then we have

$$\frac{\partial u(t, c_1^*(t))}{\partial c_1} - \lambda_{c, 2}(t) S_1(t) = 0$$

$$E_t \left[ \frac{\partial u(t + 1, c_1^*(t + 1))}{\partial c_1} S_1(t + 1) \right] - \lambda_{c, 2}(t) S_1(t) = 0$$

$$E_t \left[ \frac{\partial u(t + 1, c_1^*(t + 1))}{\partial c_1} F(t + 1, s) \right] - \lambda_{c, 2}(t) F(t, s) = 0$$

and

$$\lambda_{c, 2}(t) = \frac{\partial u(t, c_1^*(t))}{\partial c_1} \frac{1}{S_1(t)}$$

$$S_1(t) = E_t \left[ \frac{\partial u(t + 1, c_1^*(t + 1))}{\partial c_1} S_1(t + 1) \right] / \left[ \frac{S_1(t + 1)}{S_1(t)} \right]$$

$$F(t, s) = E_t \left[ \frac{\partial u(t + 1, c_1^*(t + 1))}{\partial c_1} F(t + 1, s) \right] / \left[ \frac{S_1(t + 1)}{S_1(t)} \right]$$

From Theorem 23.5 and Corollary 23.5.1 of Rockafellar (1970), the following lemma is obvious.
Lemma 5.4. There exist inverse functions to

\[
\begin{align*}
\phi_u(c_1(t)) &= -\frac{\partial u(t, c_1(t))}{\partial c_1} \\
\phi_{u, S_k}(\theta_{S_k}(t)) &= -E_t \left[ \frac{\partial V_c(t + 1, f_{S_k, I}(\theta_{S_k}(t)))}{\partial w} S_k(t + 1) \right], k = 0, 1, 2. \\
\phi_{u, F, t, s}(\theta_{F}(t, s)) &= -E_t \left[ \frac{\partial V_c(t + 1, f_{S_k, I}(\theta_{F}(t, s)))}{\partial w} F(t + 1, s) \right], s = t + 1, \ldots, T.
\end{align*}
\]

where

\[
\begin{align*}
f_{S_k, I}(\theta_{S_k}(t)) &= \theta_{S_k}(t)S_k(t + 1) + \sum_{l \neq k} \theta_{S_l}^*(t)S_l(t + 1) + \sum_{s = t + 1}^T \theta_F(t, s)F(t + 1, s) \\
f_{S_k, I}(\theta_{F}(t, s)) &= \sum_{k = 0}^2 \theta_{S_k}^*(t)S_k(t + 1) + \theta_F(t, s)F(t + 1, s) + \sum_{v \neq s} \theta_F(t, v)F(t + 1, v)
\end{align*}
\]

We denote the inverse functions \( I_{c, t}, I_{S_k, t}, \) and \( I_{F, t, s}, \) respectively.

Applying this lemma, we have

\[
c_1^*(t) = I_{c, t}(-\lambda_{c, 2}(t)S_1(t))
\]
References


