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**Abstract:** The critical capital stock is a threshold that appears in a nonconcave growth model, such that any optimal capital path from a stock level below (above) the threshold converges to a lower (higher) steady state. It explains history-dependent development and provides an implication for the achievement of sustainable development. The threshold is rarely an optimal steady state and thus it is hard to characterize. In a continuous time growth model with a convex–concave production function, we show that: a) the critical capital stock is continuous and increasing in the discount rate; b) as the discount rate increases, the critical capital stock appears from the zero stock level and disappears at a stock level between those of the maximum average and maximum marginal productivities; c) at this upper bound, the critical capital stock coalesces with the higher optimal steady state; d) once the critical capital stock disappears, the higher steady state is no longer an optimal steady state; and e) the critical capital stock at the upper bound can be arbitrarily close to either the stock level of the maximum average productivity or that of the maximum marginal productivity, depending on the curvature of the utility function.

Keywords: Continuous time growth model, convex–concave production function, critical capital stock

JEL codes: C61; D90; O41

# 1 Introduction

An aggregate growth model with a convex–concave production function is known to exhibit a complicated dynamics: depending on the initial level of capital stock, the economy may advance to a higher steady state or decline to a lower steady state. Such history dependence and polarization have been subjects in several economic research strands and the model has a wide range of applications, including in economic development (Azariadis and Drazen 1990, Askenazy and Le Van 1999, Hung, Le Van and Michel 2009, Le Van et al. 2016), firm dynamics (Davidson and Harris 1981, Hartl et al. 2004, Haunschmied et al. 2005, Wagener 2005, Caulkins et al. 2010, 2015), public policy (Brock and Dechert 1983, Caulkins et al. 2001, 2005, 2006, 2007a, 2007b, Feichtinger and Tragler 2002, Feichtinger et al. 2002), international trade (Long et al. 1997, Majumdar and Mitra 1995, Le Van et al. 2010), resource and environmental economics (Clark 1971, Dasgupta and Mäler 2003, Brock and Starrett 2003), and general theoretical studies (Majumdar and Mitra 1982, 1983, Majumdar and Nermuth 1982, Dechert and Nishimura 1983, Amir et al. 1991, Haunschmied et al. 2003, Wagener 2003, 2006, Dockner and Nishimura 2005, Kamihigashi and Roy 2006, 2007, Kiseleva and Wagener 2010). Deissenberg et al. (2004) provide a survey of this topic.

The threshold appearing in a nonconcave model is known as the critical capital stock.[1] It may be expected that the critical capital stock is an unstable steady state of the canonical system of Hamiltonian differential equations. This could be true if the model were a concave model with wealth effects (Kurz 1968, Wirl and Feichtinger 2005), in which the unstable steady state is an optimal steady state. However, the optimality is hardly expected in a nonconcave model because the production function is convex at the steady state. In fact, sufficient conditions are known under which the unstable steady state is not an optimal steady state (Dechert and Nishimura 1982, Askenazy and Le Van 1999). While the critical capital stock has crucial implications for economic development, its characterization is difficult and it has not been well investigated.

Even the existence of the critical capital stock is not fully understood. As we will show, and as can be

[1] This threshold is also known as the Skiba point or the Dechert–Nishimura–Skiba point (Haunschmied et al. 2003) because Skiba (1978) suggested its existence and Dechert and Nishimura (1983) proved that it exists in a certain range of discount rates. Clark's (1971) work on renewable resource management is potentially the earliest analysis of this critical threshold. While Dechert and Nishimura (1983) used a discrete time model, Askenazy and Le Van (1999) proved its existence in a continuous time model. See also Long et al. (1997) and Dockner and Nishimura (2005).

intuitively understood, the critical capital stock exists if and only if the two steady states of the canonical system at either side of the critical capital stock are optimal and stable. Therefore, the existence is related to the optimality of steady states. In a nonconcave model, whether a steady state is optimal is not trivial because Arrow's sufficiency theorem is not applicable. It is a basic question, but it has not been well addressed. Skiba (1978), one of the earliest studies indicating the critical capital stock, did not mention the optimality of a steady state. Wagener (2003) developed a local criterion of the heteroclinic bifurcation of the canonical system and showed that the bifurcation implies the existence of a critical capital stock. Kiseleva and Wagener (2010) made a detailed bifurcation analysis in a nonconcave model called the shallow lake model. While the validity of their analysis is clear on the phase diagrams, there is no formal proof for the optimality of the solutions of the canonical system.

It is known that an interior stable steady state is optimal if the discount rate is less than or equal to the maximum average productivity.[2] However, with a larger discount rate, the optimality is ambiguous. The only examples known are in discrete time models, which show that it may not be an optimal steady state (Majumdar and Mitra 1982) and that it can be an optimal steady state (Dechert and Nishimura 1983). The demise of the optimality of the interior stable steady state has an important implication: an unsustainable path toward the zero stock level becomes optimal for any stock level. Therefore, when it occurs is important, but the identification has not been addressed.

In this paper, we investigate these problems in a continuous time model. In a discrete time model, Akao, Kamihigashi, and Nishimura (2012) proved the continuity and monotonicity of the critical capital stock in the discount rate. We reproduce their results, but the proof is rather different as a result of the differences in Bellman equations between a continuous time model and a discrete time model. We also provide further results. The critical capital stock increases in the discount rate, and there is a discount rate at which the critical capital stock coalesces with the higher optimal steady state. With a higher discount rate, the critical capital stock vanishes. Although the steady state remains, it is no longer optimal. The upper bound of the critical capital stock can be arbitrarily close to either the stock of the maximum average productivity or that of the maximum marginal productivity. In other words, the

[2] See Dechert and Nisimura (1983, Lemma 2) in a discrete time model and Askenazy and Le Van (1999, Proposition 7) in a continuous time model.

higher optimal steady state can lose optimality when the discount rate becomes slightly larger than the maximum average productivity or it can remain optimal until the steady state of the canonical system exists. Which of these occurs depends on the intertemporal elasticity of substitution for consumption. The intuitive understanding is that with a low intertemporal elasticity, the consumption smoothing effect is strong and it prevents the optimal state path from going down toward the lower optimal steady state, against increasing discount rates.

The remainder of the paper is organized as follows. Section 2 details the model and the assumptions. Section 3 provides some preliminary results on the optimal paths. Section 4 shows the results concerning the critical capital stock. Section 5 shows the relation between the curvature of the utility function and the optimality of the upper steady state of the canonical system, using the constant intertemporal elasticity of substitution (CIES) utility function. Section 6 concludes.

# 2 Model and assumptions

Consider the following continuous time optimal growth model:

$$V^*(x_0) := \sup_{c(t)} \int_0^\infty u\left(c(t)\right) e^{-\rho t} dt \tag{2.1}$$

$$\text{subject to } \dot{x}(t) = f\left(x(t)\right) - c(t),\ c(t) \geq 0,\ x(t) \geq 0,\ x(0) = x_0 > 0 \text{ given,}$$

where $c(t)$ is the consumption path, $x(t)$ is the capital path, $x_0$ is the initial capital stock, and $\rho > 0$ is the discount rate. A path $(x(t), c(t))$ is called a feasible path from $x_0$ if it satisfies the nonnegativity condition in (2.1) and $x(t)$ is a unique solution of the state equation with the initial value $x_0$. Then, a feasible path $(x^*(t), c^*(t))$ from $x \geq 0$ is optimal if there is no feasible path $(x(t), c(t))$ from $x$ that satisfies:

$$\int_0^\infty \left[u\left(c(t)\right) - u\left(c^*(t)\right)\right] e^{-\rho t} dt > 0.$$

Throughout the paper, we use $(x^*(t), c^*(t))$ to denote an optimal path.

We make the following assumptions. Assumption 2 (b) below shows that the production function is convex–concave and the problem (2.1) is nonconcave.

**Assumption 1**: The utility function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is twice continuously differentiable on $(0, \infty)$, and satisfies $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c\to 0} u'(c) = \infty$.

**Assumption 2**: The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice continuously differentiable function with the following properties: (a) $f(0) = 0$, (b) there is an inflection point $x_I$ such that $f''(x) \gtreqless 0$ for $x \lesseqgtr x_I$, (c) $\lim_{x\to 0} f'(x) > 0$, (d) $\lim_{x\to 0} f''(x)$ exists, and (e) $\lim_{x\to\infty} f'(x) = 0$.

With these assumptions, Akao, Ishii, Kamihigashi, and Nishimura (2019) show the following:

**Proposition 2.1** *(Existence) (i) The problem (2.1) has an optimal path that is an interior path,* $(x^*(t), c^*(t)) > 0$ *for all* $t \geq 0$. *(ii) The optimal value function* $V^* : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ *is continuous and nondecreasing.*

**Proof.** See Theorem 4.1, Proposition 2.1, and Proposition 2.2 in Akao, Ishii, Kamihigashi and Nishimura (2019). ■

**Remark**: If Assumption 2 (e) is replaced with $\lim_{x\to\infty} f'(x) < 0$, which is typically used in renewable resource economics, we have the same proposition as above and the same results shown in this paper by restricting the state space to $[0, \max\{x|f(x) \geq 0\}]$. For any initial stock and any feasible path, the capital stock enters into this interval in a finite time and stays there. In this sense, this restriction is innocuous. If Assumption 2 (e) is replaced with $\lim_{x\to\infty} f'(x) > 0$, then the above proposition and the results shown in this paper are valid when the discount rate satisfies $\rho > \lim_{x\to\infty} f'(x)$.

We define the two discount rates $\rho_0$ and $\rho_I$ by:

$$\rho_0 := \lim_{x\to 0} f'(x) \text{ and } \rho_I := \max\{f'(x)|x \geq 0\}( = f'(x_I)). \tag{2.2}$$

If $\rho \in (\rho_0, \rho_I)$, there are two positive stock levels that satisfy $f'(x) = \rho$. We denote these by $x_s$ and $x^s$, respectively, where $x_s < x^s$, and we refer to them as the lower stationary capital stock and the upper stationary capital stock, respectively. They are also denoted by $x_s(\rho)$ and $x^s(\rho)$ when we highlight the fact that they are functions of $\rho$. We apply this same convention to the other variables.

The Hamiltonian $H : \mathbb{R}^3_{++} \to \mathbb{R}$ and the maximized Hamiltonian $H^* : \mathbb{R}^2_{++} \to \mathbb{R}$ associated with the problem (2.1) are defined by:

$$H(c, x, q) := u(c) + q(f(x) - c), \tag{2.3}$$

$$\text{and } H^*(x, q) := \max \{H(c, x, q)|c \geq 0\}, \tag{2.4}$$

respectively.

From Pontryagin's maximum principle, an interior optimal path is a solution of the canonical system of Hamiltonian differential equations:

$$\dot{x}(t) = \partial H^*(x(t), q(t))/\partial q = f(x(t)) - u'^{-1}(q(t)), \tag{2.5a}$$

$$\dot{q}(t) = \rho q(t) - \partial H^*(x(t), q(t))/\partial x = -[f'(x(t)) - \rho]q(t), \tag{2.5b}$$

where $u'^{-1}$ is the inverse function of $u'$, i.e., $c = u'^{-1}(q) \Longleftrightarrow u'(c) = q$.

Let:

$$\sigma(c) := -\frac{cu''(c)}{u'(c)}. \tag{2.6}$$

The following system of differential equations:

$$\dot{x}(t) = f(x(t)) - c(t), \tag{2.7a}$$

$$\dot{c}(t) = \frac{c(t)}{\sigma(c(t))}[f'(x(t)) - \rho], \tag{2.7b}$$

is equivalent to the canonical system (2.5). We refer to it as the $x$-$c$ system and call a solution of the system an $x$-$c$ path.

Let:

$$c_s := f(x_s) \text{ and } c^s := f(x^s). \tag{2.8}$$

The steady states of the canonical system and the $x$-$c$ system are $(x_s, u'(c_s))$, $(x^s, u'(c^s))$, and $(x_s, c_s)$, $(x^s, c^s)$, respectively. Corresponding to the lower and upper stationary capital stocks, these are called the lower steady state and the upper steady state of these systems.

The Jacobian of the $x$-$c$ system is given by:

$$J = \begin{bmatrix} f'(x) & -1 \\ cf''(x)/\sigma(c) & \left[d\left(c/\sigma(c)\right)/dc\right]\left[f'(x(t)) - \rho\right] \end{bmatrix}. \tag{2.9}$$

When $\rho \in (\rho_0, \rho_I)$, the eigenvalues associated with the steady states are:

$$\frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4cf''(x)/\sigma(c)}\right), \tag{2.10}$$

where $(x, c) = (x_s, c_s), (x^s, c^s)$. Thus, $(x_s, c_s)$ is unstable, whereas $(x^s, c^s)$ is saddle-point stable.

The gain function (Kamihigashi and Roy 2006, 2007) is a useful tool to characterize an optimal path. The continuous time version is defined by:

$$\gamma(x) := f(x) - \rho x. \tag{2.11}$$

When $\rho \in (\rho_0, \rho_I)$, the lower stationary capital stock $x_s$ is the local minimizer and the upper stationary capital stock $x^s$ is the local maximizer of $\gamma(x)$. See Figure 1.

<Figure 1>

We introduce the following notation. Let $\hat{\rho}$ be the discount rate that coincides with the maximum average productivity:

$$\hat{\rho} := \max\{f(x)/x | x \geq 0\}. \tag{2.12}$$

We use $\hat{x}$ to denote the capital stock of the maximum average productivity:

$$\hat{x} := \arg\max\{f(x)/x | x \geq 0\}. \tag{2.13}$$

We also define two capital stock levels, $\breve{x}$ and $\underline{x}$. $\breve{x}$ is implicitly defined by:

$$\gamma(\breve{x}) = 0, \ \gamma'(\breve{x}) > 0, \ \breve{x} > 0. \tag{2.14}$$

$\breve{x}$ exists when $\rho \in (\rho_0, \hat{\rho}]$. It holds that $\gamma(x) < 0$ for all $x \in (0, \breve{x})$, whereas $\gamma(x) > 0$ for all $x \in (\breve{x}, x^s)$. As easily verified, $\breve{x}(\rho)$ is continuous and strictly increasing. $\lim_{\rho \searrow \rho_0} \breve{x}(\rho) = 0$ and $\lim_{\rho \nearrow \hat{\rho}} \breve{x}(\rho) = x^s(\hat{\rho})$ hold.

$\underline{x}$ is implicitly defined by:

$$\gamma(\underline{x}) = \gamma(x^s), \ \underline{x} \in (0, x^s). \tag{2.15}$$

$\underline{x}$ exists when $\rho \in (\hat{\rho}, \rho_I)$. $\underline{x}$ has the property that $\gamma(x) > \gamma(x^s)$ for all $x \in (0, \underline{x})$, whereas $\gamma(x) \leq \gamma(x^s)$ for all $x \geq \underline{x}$. As easily verified, $\underline{x}(\rho)$ is continuous and strictly increasing, and it satisfies $\lim_{\rho \searrow \hat{\rho}} \underline{x}(\rho) = 0$ and $\lim_{\rho \nearrow \rho_I} \underline{x}(\rho) = x_I$. See Figure 1 for their geometry.

# 3 Optimal paths

This section shows some results regarding the optimal paths, based on which we characterize the critical capital stock. The following proposition summarizes the results that are known or easily derived.

**Proposition 3.1** *(Monotonicity and steady states)*

*(i) An optimal capital path is monotonic. A nonconstant optimal path converges to either the origin or the upper steady state.*

*(ii) In the case of mild discounting $\rho \in (0, \rho_0]$, every optimal path from $x > 0$ converges to the upper steady state, whereas in the case of heavy discounting $\rho \in (\rho_I, \infty)$, every optimal path converges to the origin.*

*(iii) When $\rho \in (\rho_0, \hat{\rho}]$, every optimal path from $x \geq \breve{x}$ converges to the upper steady state.*

*(iv) When $\rho \in (\rho_0, \rho_I]$, there is a capital stock level $x'$ such that every optimal path from $x < x'$ converges to the origin. In particular, if $\rho \in (\hat{\rho}, \rho_I)$, every optimal path from $x \leq \underline{x}$ converges to the origin.*

*(v) If the lower steady state $(x_s, c_s)$ is an optimal steady state, it is unstable.*

*(vi) The lower steady state $(x_s, c_s)$ is not an optimal steady state if:*

$$\rho^2 < 4f''(x_s)c_s/\sigma(c_s). \tag{3.1}$$

**Proof.** See Appendix A.2. ■

One thing that is not trivial as a result of the nonconcavity of the problem is the uniqueness of an optimal path. We can show the following:

**Proposition 3.2** *(Uniqueness) (i) An optimal path converging to the upper steady state is unique. (ii) An optimal path converging to the origin is unique. (iii) When the lower steady state $(x_s, c_s)$ is an optimal steady state, there is no nonconstant optimal path from $x_s$.*

**Proof.** See Appendix A.3. ■

**Remark**: This proposition does not claim that every optimal path is unique. As we will see in the next section, there are two optimal paths starting from the critical capital stock unless the lower steady state is an optimal steady state (Proposition 4.2).

The comparative statics results below play a key role in obtaining the monotonicity of the critical capital stock in the discount rates. In what follows, we compare two paths that differ in the discount rates, $\rho_i$ $(i = 1, 2)$. We refer to an optimal path when the discount rate is $\rho$ as a $\rho$-optimal path. We apply this same convention to the other paths and functions. We also introduce the terms for the ascending and descending paths. We call an $x$-$c$ path $(x^A(t), c^A(t))$, such that $\dot{x}^A(t) > 0$ for all $t \geq 0$ and $\lim_{t\to\infty} x^A(t) = x^s$, an ascending path. Similarly, we call an $x$-$c$ path $(x^D(t), c^D(t))$, such that $\dot{x}^D(t) < 0$ for all $t \geq 0$ and it converges to $(0, 0)$ or $(x^s, c^s)$, a descending path. If a descending path converges to the origin, we call it a descending path to zero and denote it by $(x^{D0}(t), c^{D0}(t))$. When it converges to the upper steady state, we call it a descending path to $x^s$ and denote it by $(x^{Ds}(t), c^{Ds}(t))$. Figure 2 illustrates the $\rho_1$- and $\rho_2$-ascending and descending paths, which are indicated by the propositions below.

<Figure 2>

**Proposition 3.3** *(Ascending and descending to $x^s$ paths) Let $\rho_1, \rho_2$ satisfy $0 < \rho_1 < \rho_2$. (i) Let $(x^A(t;\rho_1), c^A(t;\rho_1))$ and $(x^A(t;\rho_2), c^A(t;\rho_2))$ be $\rho_1$- and $\rho_2$-ascending paths from the same initial capital stock. Then, $c^A(0;\rho_1) < c^A(0;\rho_2)$. (ii) Let $(x^{Ds}(t;\rho_1), c^{Ds}(t;\rho_1))$ and $(x^{Ds}(t;\rho_2), c^{Ds}(t;\rho_2))$ be $\rho_1$- and $\rho_2$-descending paths to $x^s$ from the same initial capital stock. Then, $c^{Ds}(0;\rho_1) < c^{Ds}(0;\rho_2)$.*

**Proof.** See Appendix A.4. ■

**Proposition 3.4** *(Descending optimal paths to zero) Let $x_0 > 0$ and $\rho_1, \rho_2$ satisfy $\rho_0 < \rho_1 < \rho_2$. (i) Let $(x^{*D0}(t;\rho_1), c^{*D0}(t;\rho_1))$ and $(x^{*D0}(t;\rho_2), c^{*D0}(t;\rho_2))$ be $\rho_1$- and $\rho_2$-descending optimal paths to zero starting from the same initial capital stock $x_0$. Then, $c^{*D0}(0;\rho_1) < c^{*D0}(0;\rho_2)$. (ii) If the $\rho_1$-descending optimal path to zero starting from $x_0$ exists, then a $\rho_2$-descending path to zero starting from the same initial capital stock $x_0$ exists.*

**Proof.** See Appendix A.4. ■

**Remark**: Although an ascending path and a path descending to $x^s$ are unique, a descending path to zero may not be unique. As we require uniqueness, Proposition 3.4 restricts the statement to an optimal path.

# 4 Critical capital stock

This section investigates the critical capital stock, which is defined as follows:

**Definition** *The critical capital stock $x^C$ is a positive capital stock such that every optimal capital path from $x < x^C$ converges to $0$ and every optimal capital path from $x > x^C$ converges to the upper stationary capital stock $x^s$.*

The first result concerns the existence of the critical capital stock.

**Proposition 4.1** *(Existence and uniqueness) There exists a unique critical capital stock $x^C(\rho)$ if and only if $\rho > \rho_0$ and the upper stationary capital stock $x^s(\rho)$ is an optimal stationary capital stock.*

**Proof.** See Appendix A.5. ■

Next, we show the relation of the critical capital stock and the lower stationary capital stock $x_s$. The proposition also provides a further result regarding the uniqueness of an optimal path. Recall that Proposition 3.2 does not mention the uniqueness of an optimal path starting from the critical capital stock.

**Proposition 4.2** *(Relation with the lower stationary capital stock) (i) If $x_s$ is an optimal stationary capital stock, then it is the critical capital stock. In this case, every optimal path is unique. (ii) If $x_s$ is not an optimal stationary capital stock, then there are two optimal paths starting from the critical capital stock: One converges to the upper steady state and the other converges to the origin.*

**Proof.** See Appendix A.6. ■

**Remarks**:

1. The converse of the above proposition (i) is not true: $x^C = x_s$ does not imply that $x_s$ is an optimal stationary capital stock. From Proposition 4.3 below, the critical capital stock coincides with the lower stationary capital stock at some discount rate. However, $x_s$ cannot be an optimal stationary capital stock if the condition (3.1) in Proposition 3.1 (vi) is satisfied.

2. Nevertheless, $x_s$ can be an optimal stationary capital stock. Akao, Kamihigashi, and Nishimura (2019) exemplified a parametric growth model in which the lower steady state is optimal. Kiseleva and Wagener (2010) showed on the phase diagrams that $x_s$ can be an optimal stationary capital stock in the shallow lake model.

The following proposition is the counterpart of the main results of Akao, Kamihigashi, and Nishimura (2012, Propositions 4.1 and 4.2). Here, the monotonicity is strengthened to strict inequality: $x^C(\rho) < x^C(\rho')$ for $\rho < \rho'$.

**Proposition 4.3** *(Continuity and monotonicity) (i) There is a $\rho_H \in [\hat{\rho}, \rho_I]$ such that $x^C(\rho)$ exists if and only if $\rho_0 < \rho \leq \rho_H$. (ii) $x^C(\rho)$ is continuous and strictly increasing with $\lim_{\rho \searrow \rho_0} x^C(\rho) = 0$ and $x^C(\rho_H) = x^s(\rho_H)$.*

**Proof.** See Appendix A.7. ∎

**Remark**: From Proposition 4.1, the statement (i) above is equivalent to the following corollary. Indeed, we prove this corollary in the Appendix.

**Corollary 4.1** *There is a* $\rho_H \in [\hat{\rho}, \rho_I]$ *such that* $x^s(\rho)$ *is an optimal stationary capital stock if and only if* $\rho \leq \rho_H$.

Next, we present a result concerning the location of the critical capital stock. Recall that $\breve{x}$ and $\underline{x}$ were defined in (2.14) and (2.15), respectively.

**Proposition 4.4** *(Location of the critical capital stock) (i) If* $\rho \in (\rho_0, \hat{\rho}]$*, then* $x^C(\rho) \leq \breve{x}(\rho)$*. (ii) If* $\rho \in (\hat{\rho}, \rho_H)$ *and* $x^C(\rho)$ *exists, then* $x^C(\rho) \geq \underline{x}(\rho)$*.*

**Proof.** Proposition 3.1 (iii) and (iv), respectively, imply (i) and (ii). ∎

Figure 3 illustrates numerical examples with the production function:

$$f(x) = 10^{-3}\ln(x+1) + \frac{x^2}{4(x^2+1)},$$

for which $x_I = 0.5768$, $\hat{x} = 0.9985$, $\rho_0 = 10^{-3}$, $\hat{\rho} = 0.1257$, and $\rho_I = 0.1630$. The utility function is of the CIES type: $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$, $\sigma > 0$. Panel (a) depicts the production function. Panels (b-1) and (b-2) illustrate the case when $\sigma$ is 0.7. Panel (b-1) depicts the phase diagram when the discount rate coincides with the maximum average productivity ($\rho = \hat{\rho}$). The two vertical lines show the nullclines of $\dot{c} = 0$ that are located at the lower and upper stationary capital stocks. In this case, from Proposition 4.3 (i), the critical capital stock exists. Panel (b-2) represents the case when the discount rate coincides with the maximum marginal productivity ($\rho = \rho_I$). The vertical line is at the inflection point $x_I$. The phase diagram shows that the critical capital stock coalesces with the optimal steady state, as suggested by Proposition 4.3 (ii).

Panels (c-1)–(c-3) illustrate the case when the elasticity of the marginal utility is smaller: $\sigma = 0.3$. With respect to the discount rate, (c-1) corresponds to (b-1), and (c-3) corresponds to (b-2). The difference from Panel (b) is that when $\rho = \rho_I$, there is no longer a critical capital stock. The heteroclinic bifurcation (the coalescence of the upper stationary capital stock and the critical capital stock) occurs with a lower discount rate, which is shown in Panel (c-2).

<Figure 3>

# 5 The upper bound of the discount rate

Proposition 4.3 shows that there is an upper bound of the discount rate $\rho_H \in [\hat{\rho}, \rho_I]$ beyond which the critical capital stock and the optimal steady state do not exist. At $\rho_H$, the critical capital stock coalesces to the upper stationary capital stock that is an optimal steady state. For $\rho > \rho_H$, there is no longer a critical capital stock or an optimal interior steady state, although the upper steady state may exist. Figure 3 indicates that the level of the upper bound is affected by the curvature of the utility function. This section shows that, depending on the curvature of the utility function, the critical capital stock and the optimal steady state can survive even at a discount rate almost as high as $\rho_I$, or they can disappear even at a discount rate slightly greater than $\hat{\rho}$. To this end, this section assumes the following CIES utility function.

**Assumption 3**:

$$u(c) = \begin{cases} c^{1-\sigma}/(1-\sigma) & \text{if } \sigma > 0 \text{ and } \sigma \neq 1 \\ \ln c & \text{if } \sigma = 1 \end{cases} . \tag{5.1}$$

First, we show that if $\sigma$ is sufficiently large, for any $\rho < \rho_I$, $x^s(\rho)$ is an optimal stationary capital stock and, thus, the critical capital stock exists. Fix $\rho \in (\hat{\rho}, \rho_I)$. Consider the following piecewise linear production function $\tilde{f}(x)$:

$$\tilde{f}(x) = \begin{cases} \alpha x & \text{if } 0 \leq x < \underline{x} \\ \rho x - (\rho - \alpha)\underline{x} & \text{if } x \geq \underline{x} \end{cases} , \tag{5.2}$$

where $\underline{x}$ is defined in (2.15) and $\alpha$ is given by:

$$\alpha = f(\underline{x})/\underline{x}. \tag{5.3}$$

Note that:

$$0 < \alpha = \frac{f(\underline{x})}{\underline{x}} < \frac{f(\hat{x})}{\hat{x}} = \hat{\rho} < \rho. \tag{5.4}$$

As shown in Figure 4, $\tilde{f}(x) \geq f(x)$ with equality only if $x \in \{0, \underline{x}, x^s(\rho)\}$. From this inequality, if $x^s(\rho)$ is an optimal steady state to the problem:

$$\max_{c \geq 0} \int_0^\infty u(c) e^{-rt} dt \text{ subject to } \dot{x} = \tilde{f}(x) - c,\ x \geq 0,\ x(0) \text{ given}, \tag{5.5}$$

then $x^s(\rho)$ is also an optimal steady state to the problem (2.1).

<Figure 4>

**Lemma 5.1** *Problem (5.5) has the following closed-form solution for the optimal consumption policy:*

$$\tilde{C}(x) = \begin{cases} \beta x & \text{if } 0 \leq x \leq \underline{x} \\ \beta \underline{x} & \text{if } \underline{x} < x \leq \tilde{x}^C \\ \tilde{f}(x) & \text{if } \tilde{x}^C \leq x \end{cases}, \tag{5.6}$$

*where* $\beta = (1/\sigma)(\rho - \alpha) + \alpha$ *and* $\tilde{x}^C$ *is given by:*

$$\tilde{x}^C = \left(1 + \frac{\rho - \alpha}{\sigma \rho}\right) \underline{x}. \tag{5.7}$$

**Proof.** See Appendix A.8. ■

We define that:

$$\tilde{\sigma}(\rho) := \frac{-\gamma(x^s)}{x_s - \underline{x}} \rho^{-1}, \quad \rho \in (\hat{\rho}, \rho_I) \tag{5.8}$$

where $\gamma(x)$ is the gain function defined in (2.11).

**Proposition 5.1** *For any $\rho \in (\hat{\rho}, \rho_I)$, $x^s(\rho)$ is an optimal steady state if $\sigma \geq \tilde{\sigma}(\rho)$.*

**Proof.** Fix $\rho \in (\hat{\rho}, \rho_I)$. From (5.8) and:

$$\gamma(x^s) = \gamma(\underline{x}) = \left(\frac{f(\underline{x})}{\underline{x}} - \rho\right)\underline{x} = (\alpha - \rho)\,\underline{x},$$

we have:

$$x^s = \left(1 + \frac{\rho - \alpha}{\tilde{\sigma}\rho}\right)\underline{x}. \tag{5.9}$$

Then, by comparing (5.7) and (5.9), we see for all $\sigma \geq \tilde{\sigma}$, that $\tilde{x}^C \leq x^s$ and, thus, $x^s$ is an optimal steady state to the problem (5.5). This implies that $x^s$ is also an optimal steady state to the problem (2.1). ■

**Remark**: $\tilde{\sigma}(\rho)$ is increasing with $\tilde{\sigma}(\hat{\rho}) = 0$ and $\lim_{\rho \nearrow \rho_I} \tilde{\sigma}(\rho) = \infty$.

Next, we show that for any $\rho > \hat{\rho}$, we have $\rho_H < \rho$ if $\sigma$ is sufficiently small. This implies that the critical capital stock and the optimal upper stationary capital stock may disappear, even when the discount rate is slightly greater than $\hat{\rho}$.

For the proof, we prepare a lemma. The lemma shows that if the utility function is linear, then for any $\rho > \hat{\rho}$, $x^s(\rho)$ is not an optimal stationary capital stock. Fix $\rho > \hat{\rho}$ and let $c^M$ satisfy $c^M > f(x^s(\rho))$. Then, consider a linear utility version of the problem (2.1) with the maximum consumption $c^M > f(x^s)$:

$$\max_{c(t)} \int_0^\infty c(t)e^{-\rho t}dt \text{ subject to } \dot{x}(t) = f(x(t)) - c(t), c(t) \in [0, c^M], x(t) \geq 0, x(0) = x^s \text{ given.}$$

Let $x^M(t)$ be the capital path from $x^s(\rho)$ induced by the following most rapid extinction policy:

$$c = \begin{cases} c^M & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}. \tag{5.10}$$

Let us define the total utility associated with $x^M(t)$ by:

$$V_L(c^M) := \int_0^{T^*(c^M)} c^M e^{-\rho t} dt, \tag{5.11}$$

where $T^*(c^M)$ represents the first time that $x^M(t) = 0$.

**Lemma 5.2** *Let $\rho > \hat{\rho}$. When $c^M$ is sufficiently large:*

$$V_L(c^M) > \int_0^{\infty} f(x^s(\rho)) e^{-\rho t} dt. \tag{5.12}$$

**Proof.** See Appendix A.9. ■

**Proposition 5.2** *For any $\rho > \hat{\rho}$, there is a $\sigma^M(\rho)$ such that $x^s(\rho)$ is not an optimal steady state if $\sigma \leq \sigma^M(\rho)$.*

**Proof.** Fix $\rho > \hat{\rho}$. We choose a value of $c^M$ such that (5.12) in Lemma 5.2 holds. Note that:

$$\int_0^{T^*(c^M)} u(c^M) e^{-\rho t} dt = \frac{\left(1 - \exp\left(-\rho T^*(c^M)\right)\right)}{\rho} \frac{\left(c^M\right)^{1-\sigma}}{1-\sigma} \to V_L(c^M) \quad (\sigma \to 0)$$

and:

$$\int_0^{\infty} u(f(x^s(\rho))) e^{-\rho t} dt = \frac{\left(f(x^s(\rho))\right)^{1-\sigma}}{\rho\,(1-\sigma)} \to \int_0^{\infty} f(x^s(\rho)) e^{-\rho t} dt \quad (\sigma \to 0).$$

Then:

$$\int_0^{T^*(c^M)} u(c^M) e^{-\rho t} dt - \int_0^{\infty} u(f(x^s(\rho))) e^{-\rho t} dt \to V_L(c^M) - \int_0^{\infty} f(x^s(\rho)) e^{-\rho t} dt > 0$$

as $\sigma \to 0$. Therefore, if we choose a sufficiently small $\sigma^M(\rho) > 0$, then $x^s(\rho)$ is not an optimal steady state for $\sigma < \sigma^M(\rho)$. ■

# 6 Concluding remarks

A convex–concave production function implies that the production technology exhibits increasing returns to scale (IRS) for a small stock level. This assumption is plausible when, for example, we consider a process of economic development in which the beginning stages require a large initial investment in infrastructure, which implies IRS (Le Van et al. 2016). In a renewable resource management framework, there are nonconvex biological properties such as the Allee effect of the population dynamics (Clark 1971).[3] As an example from general economic phenomena, Weitzman (1982) argued that persistent involuntary unemployment can be explained with IRS, whereas the logic of constant returns to scale must imply full employment. These examples indicate that IRS for a small stock level is ubiquitous, and we always have a chance to take a path toward a lower steady state for the reason that it is optimal under a certain criterion. However, it may be undesirable from the perspective of other value judgments, involving, for example, sustainability and intergenerational equity criteria. A theoretical inquiry into the critical capital stock could provide the knowledge to avoid such a rational but undesirable path and provide insight as to why such a path has been experienced in history.

# A Appendix: Proofs

## A.1 Preliminaries

Before proceeding to the proofs, we prepare some lemmas and their corollaries. The first lemma and its corollary are the results from the gain function.

**Lemma A.1** *Let $(x(t), c(t))$ be a nonconstant feasible path such that $\gamma(x(t)) \leq \gamma(x(0))$ for all $t > 0$. Then, the constant path $(x'(t), c'(t)) = (x(0), f(x(0)))$ dominates $(x(t), c(t))$:*

$$\int_0^\infty u(c(t))e^{-\rho t}dt < \int_0^\infty u\left[f(x(0))\right]e^{-\rho t}dt.$$

[3] See also the papers collected in the special issue of Environment and Resource Economics: The Economics of Non-Convex Ecosystems.[OLE9]

**Proof.**

$$
\begin{aligned}
\int_0^\infty u(c(t))e^{-\rho t}dt &< \rho^{-1}u\left(\int_0^\infty \rho c(t)e^{-\rho t}dt\right) \\
&= \rho^{-1}u\left(\int_0^\infty \rho\gamma(x(t))e^{-\rho t}dt + \rho x(0)\right) \\
&\leq \rho^{-1}u\left(\int_0^\infty \rho\gamma(x(0))e^{-\rho t}dt + \rho x(0)\right) \\
&= \frac{u\left[f(x(0))\right]}{\rho} = \int_0^\infty u\left[f(x(0))\right]e^{-\rho t}dt, \qquad \text{(A.1)}
\end{aligned}
$$

where the first line follows from Jensen's inequality, the second line from the integration by parts, and the third line from $\gamma(x(t)) \leq \gamma(x(0))$. ■

**Corollary A.1** *If* $x = \arg\max\{\gamma(y)|y \in [0, x]\}$*, the optimal capital path starting from* $x$ *monotonically converges to a capital stock that is greater than or equal to* $x$*. Similarly, if* $x = \arg\max\{\gamma(y)|y \in [x, \infty)\}$*, the optimal capital path starting from* $x$ *monotonically converges to a capital stock that is less than or equal to* $x$*.*

**Proof.** First, we prove the monotonicity. Assume that there is an optimal path for which the capital path is not monotonic. We assume that $t_1 \geq 0$ and $t_2 > t_1$ such that $x^*(t_1) = x^*(t_2)$ and $\dot{x}^*(t_1)\dot{x}^*(t_2) < 0$. Let $\bar{t} = \arg\max\{x(t)|t \in [t_1, t_2]\}$. Consider the problem (2.1) with the initial stock $x^*(\bar{t})$. By the autonomous nature of the problem, the following period $t_2 - t_1$ capital path should be optimal:

$$
x(t) = x^*\left(t + \bar{t} - \left\lfloor \frac{t + \bar{t} - t_1}{t_2 - t_1} \right\rfloor (t_2 - t_1)\right).
$$

($\lfloor\cdot\rfloor$ is the floor function.) However, this path is dominated by the path remaining at $x^*(\bar{t})$ from Lemma A.1. For the other claims, note that an optimal capital path starting from $x = \arg\max\{\gamma(y)|y \in [0, x]\}$ should satisfy $\dot{x}^*(0) \geq 0$ and, similarly, an optimal capital path starting from $x = \arg\max\{\gamma(y)|y \in [x, \infty)\}$ should satisfy $\dot{x}^*(0) \leq 0$ from Lemma A.1. ■

**Remark**: The monotonicity of an optimal capital path can be proved by using the continuity of an optimal consumption path. (See Askenazy and Le Van, 1999, Proposition 4, and the literature cited therein.) In contrast to the existing proof, the above proof does not rely on the continuity.

The following lemma and its corollary are the results from the welfare implication of the maximized Hamiltonian along an $x$-$c$ path, which originated in Weitzman (1976). Note that this lemma covers an interior feasible path that is not necessarily an optimal path.

**Lemma A.2** *Let* $(x(t), c(t))$ *be an* $x$-$c$ *path that converges. Then:*

$$\int_0^\infty u(c(t))e^{-\rho t}dt = \rho^{-1}H^*(x(0), u'(c(0))). \tag{A.2}$$

**Proof.** As shown in Davidson and Harris (1981, Appendix), (A.2) holds if the following terminal condition holds:[4]

$$\lim_{t\to\infty} H^*(x(t), u'(c(0)))e^{-\rho t} = 0. \tag{A.3}$$

(A.3) holds if the path converges to an interior steady state. Then, assume that the path converges to $(0,0)$. We denote the associated costate by $q(t) = u'(c(t))$. From the canonical system (2.5), we have:

$$q(t) = q(0)\exp\left(\int_0^t \rho - f'\left(x(s)\right)ds\right). \tag{A.4}$$

Then, as $\dot{x}(t) = f[x(t)] - c(t) \to 0$ as $t \to \infty$, we have:

$$\begin{aligned}\lim_{t\to\infty} H^*[x(t), q(t)]e^{-\rho t} &= \lim_{t\to\infty}\left\{u\left(c(t)\right)e^{-\rho t} + q(0)\dot{x}(t)\exp\left(-\int_0^t f'\left(x(s)\right)ds\right)\right\}\\ &= \lim_{t\to\infty} u\left(c(t)\right)e^{-\rho t}. \end{aligned}\tag{A.5}$$

Thus, the statement is true if:

$$\lim_{t\to\infty} u\left(c(t)\right)e^{-\rho t} = 0. \tag{A.6}$$

[4] If the path is an optimal path, the terminal condition (A.3) always holds from Michel (1982, Theorem).

(A.6) holds if $u$ is bounded from below. Then, assume that $\lim_{c\to 0} u(c) = -\infty$ and choose a sufficiently small $x(0)$ such that $u(c(t)) < 0$ for all $t \geq 0$. Then, with an arbitrarily chosen $\hat{c} > 0$, we have:

$$
\begin{aligned}
0 &> u(c(t))\, e^{-\rho t} \\
&\geq \{u'(c(t))\,(c(t) - \hat{c}) + u(\hat{c})\} e^{-\rho t} \\
&= [q(t)(c(t) - \hat{c}) + u(\hat{c})]\, e^{-\rho t} \\
&= \left[q(0) \exp\left(\int_0^t \rho - f'(x(s)\, ds\right)(c(t) - \hat{c}) + u(\hat{c})\right] e^{-\rho t} \\
&= q(0) \exp\left(-\int_0^t f'(x(s)\, ds\right)(c(t) - \hat{c}) + u(\hat{c}) e^{-\rho t} \to 0 \text{ as } t \to \infty.
\end{aligned}
$$

■

**Corollary A.2** *Consider two x-c paths,* $(x(t), c(t))$ *and* $(x'(t), c'(t))$*, such that* $x(0) = x'(0)$ *and consider that they converge. If:*

$$c(0) > c'(0) \geq f(x(0)) \text{ or } c(0) < c'(0) \leq f(x(0)), \tag{A.7}$$

*then:*

$$\int_0^\infty u(c(t)) e^{-\rho t} dt > \int_0^\infty u(c'(t)) e^{-\rho t} dt. \tag{A.8}$$

**Proof.** By Lemma A.2, (A.8) is equivalent to:

$$H^*(x(0), u'(c(0))) > H^*(x(0), u'(c'(0))).$$

This inequality holds if (A.7) is satisfied because $H^*(x, q)$ as a function of $q$ is strictly convex and attains the global minimum at $q = u'(f(x))$. ■

## A.2 Proof of Proposition 3.1

The statements (i), (ii), (iii), and the second part of (iv) follow from Corollary A.1. Note that for (i), the gain function can have only two peaks at $x = 0$ and $x^s$. For (ii), when $\rho \in (0, \rho_0]$, the gain function has only a single peak at $x^s$. For (iii), when $\rho \in (\rho_0, \hat{\rho}]$, $x = \arg\max\{\gamma(y) | y \in [0, x]\}$ for $x \in [\breve{x}, x^s]$ and

$x = \arg\max\{\gamma(y)|y \in [x,\infty)\}$ for $x \geq x^s$. There is no stock level on $[\breve{x},\infty)\backslash\{x^s\}$ that satisfies $f'(x) = \rho$. For the first part of (iv), see Dockner and Nishimura (2005, Lemma 4). For the second part of (iv), when $\rho \in (\hat{\rho},\rho_I)$, for $x \leq \underline{x}$, $x = \arg\max\{\gamma(y)|y \in [x,\infty)\}$ and there is no stock level on $(0,\underline{x}]$ that satisfies $f'(x) = \rho$. (v) It follows from the eigenvalues in (2.10). For (vi), we prove the contraposition. If $(x_s, c_s)$ is an optimal steady state, from Proposition 3.1 (i) and (v), an optimal capital path is monotonic and, from Corollary A.2, its $\alpha$-limit point of the optimal $x$-$c$ path is $(x_s, c_s)$. This implies that the eigenvalues of the Jacobian of the $x$-$c$ system at $(x_s, c_s)$ should be real numbers. That is, $\rho^2 \geq 4f''(x_s)c_s/\sigma\,(c_s)$. ■

**Remark**: Askenazy and Le Van (1999, Proposition 10) showed another sufficient condition under which the lower steady state is not an optimal steady state:[5]

$$\rho^2 < f''(x_s)c_s/\sigma(c_s). \tag{A.9}$$

It is easily verified that (A.9) implies (3.1).

## A.3 Proof of Proposition 3.2

(i) From (2.10), $(x^s, c^s)$ is a saddle point. An optimal path converging to $(x^s, c^s)$ must be on the one-dimensional stable manifold of the $x$-$c$ system and thus, it is unique. (ii) Assume that there are two different optimal paths converging to the origin. They should start from the same initial stock $x^*(0)$ and the different initial consumptions $c^*(0)$, $c^{*\prime}(0)$ such that $c^*(0) > c^{*\prime}(0) > f(x^*(0))$. Then, from Corollary A.2, the path starting from $(x^*(0), c^*(0))$ dominates the other path. This is a contradiction. (iii) Assume that $x_s$ is an optimal stationary capital stock and there is a nonconstant optimal path starting from $x_s$. As the initial consumption is not equal to $f(x_s)$, Corollary A.2 implies that the nonconstant path dominates the optimal constant path. This is a contradiction. ■

[5] Also see Dechert and Nishimura (1983, Lemma 4).

## A.4 Proof of Propositions 3.3 and 3.4

As in the main text, let $\left(x^A(t), c^A(t)\right)$ be an ascending path and $\left(x^D(t), c^D(t)\right)$ be a descending path. We define $D^A$ and $D^D$, associated with these paths, as follows:

$$D^A := \{(x, c) \in \mathbb{R}^2_{++} | f(x) - c > 0\},$$

$$D^D := \{(x, c) \in \mathbb{R}^2_{++} | f(x) - c < 0\}.$$

In addition, we define two functions $\xi^A : D^A \times (\rho_0, \rho_I) \to \mathbb{R}$ and $\xi^D : D^D \times (\rho_0, \rho_I) \to \mathbb{R}$ by:

$$\xi^j(x, c; \rho) := \frac{c[f'(x) - \rho]}{\sigma(c)\,[f(x) - c]}, \quad (x, c) \in D^j,\ j = A, D.$$

$\xi^j$ $(j = A, D)$ give the gradient of the vector field of the $x$-$c$ system at $(x, c)$ in the domain $D^j$ $(j = A, D)$. They satisfy:

$$\frac{\partial \xi^A(x, c; \rho)}{\partial \rho} < 0 \text{ and } \frac{\partial \xi^D(x, c; \rho)}{\partial \rho} > 0. \tag{A.10}$$

Let $\inf x^A(\rho)$ be the infimum of the capital stocks from which a $\rho$-ascending path $(x^A(t; \rho), c^A(t; \rho))$ starts:

$$\inf x^A(\rho) := \inf\{x \in \mathbb{R}_+ |\text{ a } \rho\text{-ascending path starting from } x \text{ exists}\}. \tag{A.11}$$

We define the subset of the $x$-$c$ plane as:

$$epi^A(\rho) := \{(x, c) \in \left(\inf x^A(\rho), x^s(\rho)\right) \times \mathbb{R}_+ | \exists t \in \mathbb{R} : x = x^A(t; \rho) \text{ and } c > c^A(t; \rho)\}. \tag{A.12}$$

Similarly, let $\left(x^{D0}(t; \rho), c^{D0}(t; \rho)\right)$ denote a $\rho$-descending path to zero and $\left(x^{Ds}(t; \rho), c^{Ds}(t; \rho)\right)$ denote a

$\rho$-descending path to $x^s$. We define $\sup x^{D0}(\rho)$ and $\sup x^{Ds}(\rho)$ as:

$$\sup x^{D0}(\rho) := \sup\{x \in \mathbb{R}_+ | \text{ a } \rho\text{-descending path to zero from } x \text{ exists}\}, \tag{A.13a}$$

$$\sup x^{Ds}(\rho) := \sup\{x \in \mathbb{R}_+ | \text{ a } \rho\text{-descending path to } x^s \text{ from } x \text{ exists}\}. \tag{A.13b}$$

In addition, we define a subset of the $x$-$c$ plane as:

$$hyp^{Ds}(\rho) := \{(x,c) \in \big(x^s(\rho), \sup x^{Ds}(\rho)\big) \times \mathbb{R}_+ | \exists t \in \mathbb{R} : x = x^{Ds}(t;\rho) \text{ and } c < c^{Ds}(t;\rho)\}. \tag{A.14}$$

**Proof of Proposition 3.3.** (i) Let $(x^A(t;\rho_1), c^A(t;\rho_1))$ and $(x^A(t;\rho_2), c^A(t;\rho_2))$ be, respectively, $\rho_1$- and $\rho_2$-ascending paths starting from the same initial capital stock $x_0 > 0$. Assume that $c^A(0;\rho_1) \geq c^A(0;\rho_2)$. From (A.10), $\rho_1 < \rho_2$ implies that $\big((x^A(t;\rho_2), c^A(t;\rho_2)\big) \notin epi^A(\rho_1)$ for a small $t \geq 0$. On the other hand, as $x^s(\rho_1) > x^s(\rho_2) > x_0$, $(x^s(\rho_2), c^s(\rho_2)) \in epi^A(\rho_1)$. In order for the $\rho_2$-ascending path to enter $epi^A(\rho_1)$, there must be a crossing point:

$$(x_c, c_c) := (x^A(t;\rho_1), c^A(t;\rho_1)) = (x^A(t';\rho_2), c^A(t';\rho_2))$$

at some $t \geq 0$ and $t' \geq 0$ and it must satisfy:

$$\xi^A(x_c, c_c; \rho_1) \leq \xi^A(x_c, c_c; \rho_2). \tag{A.15}$$

However, this inequality contradicts (A.10), and we conclude that $\rho_1 < \rho_2$ implies that $c^A(t;\rho_1) < c^A(t;\rho_2)$.
(ii) The proof for a descending path to $x^s$ is similar to the proof for an ascending path. Consider the $\rho_1$- and $\rho_2$-descending paths to $x^s$ starting from $x_0$. Assume that $c^{Ds}(0;\rho_1) \geq c^{Ds}(0;\rho_2)$. From (A.10), $\big((x^{Ds}(t;\rho_1), c^{Ds}(t;\rho_1)\big) \notin hyp^{Ds}(\rho_2)$ for a small $t \geq 0$, whereas $(x^s(\rho_1), c^s(\rho_1)) \in hyp^{Ds}(\rho_2)$. In order for

the $\rho_1$-descending path to enter $hyp^{Ds}(\rho_2)$, there must be a crossing point:

$$(x_c, c_c) := (x^{Ds}(t; \rho_1), c^A(t; \rho_1)) = (x^{Ds}(t'; \rho_2), c^A(t'; \rho_2))$$

at some $t \geq 0$ and $t' \geq 0$ and it must satisfy:

$$\xi^D(x_c, c_c; \rho_1) \geq \xi^D(x_c, c_c; \rho_2).$$

However, this inequality contradicts (A.10) and thus $c^{Ds}(0; \rho_1) < c^{Ds}(0; \rho_2)$. ■

**Proof of Proposition 3.4.** (i) Consider the $\rho_1$- and $\rho_2$-descending optimal paths to zero starting from the same initial stock $x_0 > 0$. We denote them by $\left(x^{*D0}(t; \rho_i), c^{*D0}(t; \rho_i)\right)$, $i = 1, 2$. Assume that $c^{*D0}(0; \rho_1) > c^{*D0}(0; \rho_2)$. Consider the $\rho_2$-$x$-$y$ path starting from $\left(x_0, c^{*D0}(t; \rho_1)\right)$ and denote it by $(x'(t; \rho_2), c'(t; \rho_2))$. From (A.10), $(x'(t; \rho_2), c'(t; \rho_2))$ and $\left(x^{*D0}(t; \rho_1), c^{*D0}(t; \rho_1)\right)$ do not cross when the capital stock is in $(0, x_0)$. Also, $(x'(t; \rho_2), c'(t; \rho_2))$ and $\left(x^{*D0}(t; \rho_2), c^{*D0}(t; \rho_2)\right)$ do not cross when the capital stock is in $(0, x_0)$ because they are the solutions of the same system of differential equations with different initial values. This is the situation where the orbit of $(x'(t; \rho_2), c'(t; \rho_2))$ lies between the orbits of $\left(x^{*D0}(t; \rho_i), c^{*D0}(t; \rho_i)\right)$, $(i = 1, 2)$ when the capital stock is in $(0, x_0)$. From the definition of the descending path to zero, the following holds:

$$\lim_{t \to \infty} \left(x^{*D0}(t; \rho_1), c^{*D0}(t; \rho_1)\right) = \lim_{t \to \infty} \left(x^{*D0}(t; \rho_2), c^{*D0}(t; \rho_2)\right) = (0, 0).$$

This also implies that $\lim_{t \to \infty} (x'(t; \rho_2), c'(t; \rho_2)) = (0, 0)$. However, if so, from Corollary A.2, $(x'(t; \rho_2), c'(t; \rho_2))$ dominates the optimal path $\left(x^{*D0}(t; \rho_2), c^{*D0}(t; \rho_2)\right)$, which is a contradiction. In the case that $c^{*D0}(0; \rho_1) = c^{*D0}(0; \rho_2)$, there are time points $t_1 > 0$ and $t_2 > 0$ such that $x^{*D0}(t_1; \rho_1) = x^{*D0}(t_2; \rho_2) := x_0'$ and $c^{*D0}(t_1; \rho_1) > c^{*D0}(t_2; \rho_2)$ from (A.10). By taking $x_0'$ as the initial capital stock, we reach the same contradiction. Therefore, $c^{*D0}(0; \rho_1) > c^{*D0}(0; \rho_2)$.

(ii) As $\rho_2 > \rho_1 > \rho_0$, both the $\rho_1$- and $\rho_2$-descending optimal paths to zero exist if a sufficiently small $x_0$

is taken as the initial stock, from Proposition 3.1. We denote these paths by $\big(x^{*D0}(t; x_0, \rho_i), c^{*D0}(t; x_0, \rho_i)\big)$, $i = 1, 2$. From (i) of this Lemma, we have:

$$c^{*D0}(0; x_0, \rho_2) > c^{*D0}(0; x_0, \rho_1). \tag{A.16}$$

Consider the extended paths of these optimal paths $\big(x^{D0}(t; x_0, \rho_i), c^{D0}(t; x_0, \rho_i)\big)$, $i = 1, 2$. From part (i) of the lemma, these paths do not cross. Therefore, for any $t_1, t_2 \in \mathbb{R}$ such that $x^{D0}(t_1; x_0, \rho_1) = x^{D0}(t_2; x_0, \rho_2) < \sup x^{D0}(\rho_1)$, the following holds:

$$c^{D0}(t_2; x_0, \rho_2) > c^{D0}(t_1; x_0, \rho_1) \geq f(x^{D0}(t_1; x_0, \rho_1)). \tag{A.17}$$

This shows that $\sup x^{D0}(\rho_1) \geq \sup x^{D0}(\rho_2)$. ■

**Remark**: In a discrete time model, the monotonicity of optimal consumptions has been shown by Amir et al. (1991, Theorem 5.5. (d)). Their proof utilizes the supermodularity of the optimal value function, which is not applicable to the present model because of the difference in the Bellman equations between a discrete time model and a continuous time model. For the proof, we utilize the property of the vector fields of the $x$-$c$ system. As a result, our results in Proposition 3.3 are not limited to an optimal path.

## A.5 Proofs of Propositions 4.1

If part of the proof: assume that $\rho > \rho_0$ and $x^s(\rho)$ is an optimal stationary capital stock. From Propositions 3.1 (i) and 3.2 (i) and (ii) (relating to monotonicity and uniqueness), the state space $[0, \infty)$ is partitioned into intervals $Y$ and $Z$ such that every optimal capital path starting from $y \in Y$ converges to 0, and every optimal capital path starting from $z \in Z$ converges to $x^s$. Note that, by construction, $\sup Y \leq \inf Z$, but $\sup Y < \inf Z$ is ruled out because $x \in (\sup Y, \inf Z)$ cannot be an optimal stationary capital stock that must satisfy $f'(x) = \rho$. Also note that $\sup Y > 0$ from Proposition 3.1 (iv). Then, $x^C(\rho) = \sup Y = \inf Z \in (0, x^s(\rho)]$ is a unique critical capital stock. Only if part of the proof: Assume that $x^s(\rho)$ is not an optimal steady state or $\rho \leq \rho_0$. Then, in the former case, every nonconstant optimal path converges to

0 from Proposition 3.1 (i), whereas in the latter case, every optimal path starting from a positive capital stock converges to $x^s(\rho)$ from Proposition 3.1 (ii). Thus, in both cases, $x^C(\rho)$ does not exist. ■

## A.6 Proof of Proposition 4.2

(i) Assume that there is an ascending path $\big(x^A(t), c^A(t)\big)$ starting from $x \leq x_s$. Then, there is a $t' \geq 0$ such that $x^A(t) = x_s$ and $c^A(t) < f(x_s)$. From Corollary A.2, this contradicts that $x_s$ is an optimal stationary capital stock. A similar argument rules out that there is a descending path to zero starting from $x \geq x_s$. Therefore, there is no nonconstant optimal capital path from $x_s$ and every optimal path starting from $x < x_s$ converges to zero, and every optimal path starting from $x > x_s$ converges to zero, i.e., $x_s$ is the critical capital stock. The uniqueness follows from Proposition 3.2.

(ii) Take $x_0^D \in (0, x^C)$ and let $\big(x^{D0}(t), c^{D0}(t)\big)$ be the descending path to zero starting from $x_0^D$. Note that there is a $t^D \in \mathbb{R}_- \cup \{-\infty\}$ such that $\big(x^{D0}(t), c^{D0}(t)\big)$ is an optimal path for $t > t^D$ and $\lim_{t \searrow t^D} x^{D0}(t) = x^C$. Similarly, take $x_0^A \in (x^C, x^s)$ and let $\big(x^A(t), c^A(t)\big)$ be the ascending path starting from $x_0^A$. Then, there is a $t^A \in \mathbb{R}_{-}\{-\infty\}$ such that $\big(x^A(t), c^A(t)\big)$ is an optimal path for $t > t^A$ and $\lim_{t \searrow t^A} x^A(t) = x^C$. From Proposition 2.1 (ii) and Lemma A.2, we have:

$$\begin{aligned}\lim_{x \nearrow x^C} V^*(x) &= \rho^{-1} H^*(x^C, u'(\lim_{t \searrow t^D} c^{D0}(t))) \\ &= \rho^{-1} H^*(x^C, u'(\lim_{t \searrow t^A} c^A(t))) = \lim_{x \searrow x^C} V^*(x).\end{aligned}$$

By definition, $c^{D0}(t) > f(x^{D0}(t))$ for $t \in (t^D, \infty)$ and $c^A(t) < f(x^A(t))$ for $t \in (t^A, \infty)$. Therefore, from Corollary A.2, either of the following holds:

$$c^{D0}(t^D) > f(x^C) > c^A(t^A),$$

or

$$\lim_{t \searrow t^D} c^{D0}(t)) = f(x^C) = \lim_{t \searrow t^D} c^{D0}(t)).$$

However, the latter is ruled out because this is the case in which:

$$V^*(x^C) = \rho^{-1} H^*(x^C, f(x^C)),$$

i.e., $x^C$ is an optimal stationary capital and $x^C = x_s$. In the former case, there are two optimal paths starting from $x^C$: one is a descending path with the initial value $\left(x^C, c^{D0}(t^D)\right)$, and the other is an ascending path with the initial value from $\left(x^C, c^A(t^A)\right)$. ■

## A.7 Proof of Proposition 4.3

The proof uses the following five lemmas. The first lemma concerns the optimality of $x^s$.

**Lemma A.3** *There exists a discount rate $\rho_H \in [\hat{\rho}, \rho_I]$ such that $x^s(\rho)$ is (is not) an optimal stationary capital stock if $\rho < \rho_H$ $(\rho > \rho_H)$.*

**Proof.** If $x^s(\rho)$ is not an optimal stationary capital stock, the optimal path starting from $x^s(\rho)$ is a descending optimal path to zero. Then, from Proposition 3.4 (ii), for all $\rho' > \rho$, there is a descending path to zero from $x^s(\rho)$. As $x^s(\rho') < x^s(\rho)$, this path contains a descending path to zero from $x^s(\rho')$, and, from Corollary A.2, $x^s(\rho')$ is not an optimal stationary capital stock. $\rho_H$ is defined by:

$$\rho_H = \inf\{\rho > 0 | x^s(\rho) \text{ is not an optimal stationary capital stock}\}.$$

Note that from Proposition 3.1 (iii), $\rho_H \in [\hat{\rho}, \rho_I]$. ■

**Remark**: In Lemma A.3, it is not clear whether $x^s(\rho_H)$ is an optimal steady state. To clarify this, we need a property of the critical capital stock.

**Lemma A.4** *Let $\rho_1$ and $\rho_2$ satisfy $\rho_0 < \rho_1 < \rho_2 < \rho_H$, where $\rho_H$ is the upper bound of discount rates defined in Proposition A.3. Then:*

$$x^C(\rho_1) < x^C(\rho_2). \tag{A.18}$$

**Proof.** The proof is divided into two cases, in which $x_s(\rho_1)$ is and is not an optimal stationary capital stock.

First, we consider the case in which $x_s(\rho_1)$ is not an optimal stationary capital stock. In this case, from Proposition 4.2 (ii), we have the $\rho_1$-ascending and descending optimal paths $\left(x^{A*}(t;\rho_1), c^{A*}(t;\rho_1)\right)$, $\left(x^{D*}(t;\rho_1), c^{D*}(t;\rho_1)\right)$ starting from $x_1^{A*}(0) = x_1^{D*}(0) = x^C(\rho_1)$. Then, from Proposition 3.4 (ii), there is the $\rho_2$-descending path to zero $\left(x^D(t;\rho_2), c^D(t;\rho_2)\right)$ from $x^D(0;\rho_2) = x^C(\rho_1)$. If the $\rho_2$-ascending path $\left(x^A(t;\rho_2), c^A(t;\rho_2)\right)$ from $x^C(\rho_1)$ does not exist, then this path is optimal and $x^C(\rho_1) < x^C(\rho_2)$. If it exists, from Propositions 3.3 and 3.4:

$$c^{A*}(0;\rho_1) < c^A(0;\rho_2) \leq f(x^C(\rho_1)) < c^{D*}(0;\rho_1) < c^D(0;\rho_2).$$

Then, from Lemma A.2, we have:

$$\begin{aligned}
&\rho_2 \int_0^\infty u\left(c^A(t;\rho_2)\right) e^{-\rho_2} dt \\
&= H^*\left(x^C(\rho_1), u'(c^A(0;\rho_2))\right) < H^*\left(x^C(\rho_1), u'(c^{*A}(0;\rho_1))\right) \\
&= H^*\left(x^C(\rho_1), u'(c^{*D}(0;\rho_1))\right) < H^*\left(x^C(\rho_1), u'(c^D(0;\rho_2))\right) = \rho_2 \int_0^\infty u\left(c^D(t;\rho_2)\right) e^{-\rho_2} dt,
\end{aligned}$$

where the inequalities follow from the fact that $H^*(x,q)$ is strictly concave in $q$ and attains the minimum at $q = u'(f(x))$. As the descending path dominates the unique ascending path, we have $x^C(\rho_2) > x^C(\rho_1)$.

Next, consider the case in which $x_s(\rho_1)$ is an optimal steady state. This is the case where $x_s(\rho_1) = x^C(\rho_1) = \inf x^A(\rho_1)$. Assume that $x^C(\rho_2) \leq x^C(\rho_1)$. Since $\inf x^A(\rho_2) \geq \inf x^A(\rho_1)$ by Proposition 3.4 (i), it must satisfy $x^C(\rho_2) = x^C(\rho_1) = \inf x^A(\rho_2)$. But this implies $x^C(\rho_2) = x_s(\rho_2)$, and, thus, $x_s(\rho_1) = x_s(\rho_2)$, which is a contradiction. ■

Next, we introduce ascending and descending value functions and show that they, as well as the optimal value function, are continuous in the discount rate. These results are used below to show the coalescence of the critical capital stock and the optimal upper stationary capital at the upper bound of the discount rate $\rho_H$ (Lemma A.6) and the continuity of the critical capital stock in the discount rate (Lemma A.7).

Take an arbitrarily large $\bar{x}$ such that $0 < f'(\bar{x}) < \rho_0$ and restrict the state space to $[0, \bar{x}]$. Note that if $\rho \geq \rho_0$, any optimal capital path enters $[0, \bar{x}]$ in a finite time and stays in $[0, \bar{x}]$ from Proposition 3.1. Therefore, this restriction does not affect an optimal path starting from $x \in \mathbb{R}_+$.[6] Also, take $\bar{\rho} > \rho_I$. We use $(x^*(t; x, \rho), c^*(t; x, \rho))$ to denote the optimal path starting from $x \in [0, \bar{x}]$ when the discount rate is $\rho \in [\rho_0, \bar{\rho}]$. We choose the consumption level $\bar{c}$ that satisfies:

$$\bar{c} > \max_{\rho \in (0, \bar{\rho}], x \in [0, \bar{x}], t \in [0, \infty)} c^*(t; x, \rho). \tag{A.19}$$

$\bar{c} < \infty$ follows from the facts that an optimal path lies on a stable manifold of a steady state and the optimal consumption is increasing in the discount rate. We modify the problem (2.1) by imposing $c(t) \in [0, \bar{c}]$ and standardizing the utility by:

$$\bar{u}(c) := u(c) - u(\bar{c}).$$

The associated optimal value function $\bar{V}^* : [0, \bar{x}] \times (0, \bar{\rho}] \to \mathbb{R}_- \cup \{-\infty\}$ is defined by:

$$\bar{V}^*(x_0, \rho) := \max_{c(t)} \int_0^\infty \bar{u}\left(c(t)\right) e^{-\rho t} dt \tag{A.20}$$

$$\text{subject to } \dot{x}(t) = f\left(x(t)\right) - c(t),\ c(t) \in [0, \bar{c}],\ x(t) \geq 0,\ x(0) = x_0 \in [0, \bar{x}].$$

The optimal paths are the same as those in the original problem (2.1).

We also define two associated value functions. The ascending value function $\bar{V}^A : [0, \bar{x}] \times (0, \bar{\rho}] \to \mathbb{R}_- \cup \{-\infty\}$ is defined by:

$$\bar{V}^A(x_0, \rho) := \max_{c(t)} \int_0^\infty \bar{u}(c(t)) e^{-\rho t} dt \tag{A.21}$$

$$\text{subject to } \dot{x}(t) = f(x(t)) - c(t),\ c(t) \in [0, f(x(t))],\ x(0) = x_0 \in [0, \bar{x}].$$

[6] In fact, the interval that we consider in the lemmas A.6 and A.7 below is contained in any $[0, \bar{x}]$ as long as $\bar{x}$ satisfies $f'(\bar{x}) < \rho_0$.

The descending value function $\bar{V}^D : [0, \bar{x}] \times (0, \bar{\rho}] \to \mathbb{R}_- \cup \{-\infty\}$ is defined by:

$$\bar{V}^D(x_0, \rho) := \max_{c(t)} \int_0^\infty \bar{u}(c(t))e^{-\rho t}dt \tag{A.22}$$

$$\text{subject to } \dot{x}(t) = f(x(t)) - c(t),\ c(t) \in [f(x(t)), \bar{c}],\ x(0) = x_0 \in [0, \bar{x}].$$

The problems (A.21) and (A.22) satisfy the conditions in d'Albis, Gourdel, and Le Van (2008), and an optimal path exists from their Theorem 1. Also note that from the definition of the critical capital stock, the following hold:

$$\bar{V}^*(x, \rho) = \bar{V}^D(x, \rho) > \bar{V}^A(x, \rho) \text{ for } 0 < x < x^C(\rho),$$

$$\bar{V}^*(x, \rho) = \bar{V}^A(x, \rho) > \bar{V}^D(x, \rho) \text{ for } x > x^C(\rho) < x \leq x^s(\rho).$$

**Lemma A.5** *For each $x \in (0, \bar{x}]$, the three value functions, $\bar{V}^*(x, \rho), \bar{V}^A(x, \rho)$, and $\bar{V}^D(x, \rho)$, are continuous in $\rho \in (0, \bar{\rho})$.*

**Proof.** Fix $x \in (0, \bar{x}]$. First, we consider the optimal value function $\bar{V}^*$. For $0 < \rho_1 < \rho_2 < \bar{\rho}$, we have:

$$\begin{aligned}\bar{V}^*(x, \rho_2) &= \int_0^\infty \bar{u}(c^*(t, \rho_2))e^{-\rho_2 t}dt \geq \int_0^\infty \bar{u}(c^*(t, \rho_1))e^{-\rho_2 t}dt \\ &\geq \int_0^\infty \bar{u}(c^*(t, \rho_1))e^{-\rho_1 t}dt \\ &= \bar{V}^*(x, \rho_1) \geq \int_0^\infty \bar{u}(c^*(t, \rho_2))e^{-\rho_1 t}dt, \end{aligned} \tag{A.23}$$

where the inequalities in the first and third lines follow from the optimality of the $\rho_i$-consumption paths $c^*(t, \rho_i)$ for $i = 2, 1$, respectively, and the inequality in the second line follows from $\rho_1 < \rho_2$ and the nonpositivity of $\bar{u}(c)$. Then, we have:

$$\begin{aligned}\lim_{\rho_2\searrow\rho_1} |\bar{V}^*(x,\rho_2)-\bar{V}^*(x,\rho_1)| &= \lim_{\rho_2\searrow\rho_1}\left[\int_0^\infty \bar{u}(c^*(t,\rho_2))e^{-\rho_2 t}dt - \int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_1 t}dt\right]\\ &\leq \lim_{\rho_2\searrow\rho_1}\left[\int_0^\infty \bar{u}(c^*(t,\rho_2))e^{-\rho_2 t}dt - \int_0^\infty \bar{u}(c^*(t,\rho_2))e^{-\rho_1 t}dt\right]\\ &= \int_0^\infty \lim_{\rho_2\searrow\rho_1} \bar{u}(c^*(t,\rho_2))(e^{-\rho_2 t}-e^{-\rho_1 t})dt = 0, \end{aligned} \tag{A.24}$$

from the monotone convergence theorem. Similarly, for $\rho_3 \in (0,\rho_1)$:

$$\begin{aligned}\lim_{\rho_3\nearrow\rho_1} |\bar{V}^*(x,\rho_3)-\bar{V}^*(x,\rho_1)| &= \lim_{\rho_3\nearrow\rho_1}\left[\int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_1 t}dt - \int_0^\infty \bar{u}(c^*(t,\rho_3))e^{-\rho_3 t}dt\right]\\ &\leq \lim_{\rho_3\nearrow\rho_1}\left[\int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_1 t}dt - \int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_3 t}dt\right]\\ &= \int_0^\infty \lim_{\rho_3\nearrow\rho_1} \bar{u}(c^*(t,\rho_1))(e^{-\rho_1 t}-e^{-\rho_3 t})dt = 0. \end{aligned} \tag{A.25}$$

These results verify that $\bar{V}^*(x,\rho)$ is continuous in $\rho$. As we have derived this result by using only the negativity of $\bar{u}$ and the optimality of the consumption paths, the same argument is applied to the ascending and descending value functions and it is found that these value functions are also continuous in $\rho$. ■

Now, we show the coalescence of the critical capital stock and the optimal upper stationary capital stock at $\rho_H$ and the continuity of the critical capital stock.

**Lemma A.6** *$x^s(\rho_H)$ is an optimal stationary capital stock and $x^s(\rho_H) = x^C(\rho_H)$, where $\rho_H$ is defined in Proposition A.3.*

**Proof.** As $x^s(\rho_H) < x^s(\rho)$ for $\rho < \rho_H$, Proposition 2.1 (ii) implies that:

$$\rho^{-1}\bar{u}\left(f\left(x^s(\rho)\right)\right) = \bar{V}^*(x^s(\rho),\rho) \geq \bar{V}^*(x^s(\rho_H),\rho). \tag{A.26}$$

Using the continuity (Lemma A.5), by sending $\rho$ to $\rho_H$, we have $\rho_H^{-1}\bar{u}\left(f\left(x^s(\rho_H)\right)\right) \geq \bar{V}^*(x^s(\rho_H), \rho_H)$. As keeping $x^s(\rho_H)$ is feasible, we have:

$$\rho_H^{-1}\bar{u}\left(f\left(x^s(\rho_H)\right)\right) = \bar{V}^*(x^s(\rho_H), \rho_H). \tag{A.27}$$

That is, $x^s(\rho_H)$ is an optimal upper stationary capital stock. Next, we show that $x^C(\rho_H) = x^s(\rho_H)$. As $x^s(\rho)$ is decreasing and $x^C(\rho)$ is increasing (Lemma A.4), $x^C(\rho) < x^s(\rho_H) < x^s(\rho)$ holds. Assume that $x^C(\rho_H) < x^s(\rho_H)$. Then, for $z \in \left(x^C(\rho_H), x^s(\rho_H)\right)$:

$$\bar{V}^*(z, \rho_H) = \bar{V}^{*A}(z, \rho_H) > \bar{V}^{*D}(z, \rho_H), \tag{A.28}$$

from the definition of the critical capital stock. On the other hand, for $\rho > \rho_H$:

$$\bar{V}^*(z, \rho) = \bar{V}^{*D}(z, \rho) > \bar{V}^{*A}(z, \rho),$$

from the definition of $\rho_H$. These value functions are continuous in $\rho$ (Lemma A.5). By sending $\rho$ to $\rho_H$, we have:

$$\bar{V}^*(z, \rho_H) = \bar{V}^{*D}(z, \rho_H) \geq \bar{V}^{*A}(z, \rho_H). \tag{A.29}$$

This contradicts (A.28) and we conclude that $x^C(\rho_H) = x^s(\rho_H)$. ■

**Lemma A.7** *(Continuity) $x^C(\rho)$ is continuous on $(\rho_0, \rho_H]$, where $\rho_H$ is defined in Proposition A.3.*

**Proof.** We prove by contradiction. Suppose that there is $\rho' \in (\rho_0, \rho_H]$ at which $x^C(\rho)$ is discontinuous. From Lemma A.4, this is the case in which (a) $\lim_{\rho \nearrow \rho'} x^C(\rho) < x^C(\rho')$ and/or (b) $\lim_{\rho \searrow \rho'} x^C(\rho) > x^C(\rho')$. Suppose that case (a) occurs. Let $z \in (\lim_{\rho \nearrow \rho'} x^C(\rho), x^C(\rho'))$. Then:

$$\bar{V}^A(z, \rho') < \bar{V}^D(z, \rho') = \bar{V}^*(z, \rho'). \tag{A.30}$$

On the other hand, for $\rho < \rho'$, $\bar{V}^A(z,\rho) > \bar{V}^D(z,\rho)$ holds. The continuity of these value functions in $\rho$ (Lemma A.5) implies that $\bar{V}^A(z,\rho') \geq \bar{V}^D(z,\rho')$ at the limit $\rho \nearrow \rho'$, which contradicts (A.30). Thus, case (a) is ruled out. By a parallel argument, case (b) is also ruled out. ■

**Proof of Proposition 4.3**: (i) follows from Lemma A.6 and Proposition 3.4. (ii) follows from Lemma A.4 and A.7. ■

## A.8 Proof of Lemma 5.1

We verify the optimality of $\tilde{C}(x)$ by the Hamilton–Jacobi–Bellman equation. Denote by $T(x)$ the time to reach $\underline{x}$ from $x \in (\underline{x}, \tilde{x}^C)$. $T(x)$ is given by:

$$e^{-\rho T(x)} = \frac{\rho x - (\rho + \beta - \alpha)\underline{x}}{\rho \underline{x} - (\rho + \beta - \alpha)\underline{x}}. \tag{A.31}$$

The optimal value function $\tilde{V}^*(x)$ associated with the policy function (5.6) is given by:

$$\tilde{V}^*(x) = \begin{cases} \begin{cases} [\beta^{-\sigma}/(1-\sigma)]x^{1-\sigma} \text{ if } \sigma \neq 1 \\ \rho^{-1}(\ln \rho x + \rho^{-1}(\alpha - \beta)) \text{ if } \sigma = 1 \end{cases} & \text{for } 0 < x \leq \underline{x} \\ \int_0^{T(x)} u(\beta \underline{x}) e^{-\rho t} dt + \tilde{V}^*(\underline{x}) e^{-\rho T(x)} & \text{for } \underline{x} < x \leq \tilde{x}^C \\ u(\tilde{f}(x))/\rho & \text{for } \tilde{x}^C \leq x \end{cases}, \tag{A.32}$$

where

$$\beta = \alpha + \frac{\rho - \alpha}{\sigma}. \tag{A.33}$$

With $\tilde{V}^*(x)$ and $\tilde{C}(x)$, the following holds for each $x > 0$:

$$\begin{aligned} \rho \tilde{V}^*(x) &= u(\tilde{C}(x)) + \tilde{V}^{*\prime}(x)[\tilde{f}(x) - \tilde{C}(x)] \\ &\geq u(c) + \tilde{V}^{*\prime}(x)[\tilde{f}(x) - c] \text{ for all } c \geq 0. \end{aligned} \tag{A.34}$$

Let $(\tilde{x}(t), \tilde{c}(t))$ be a feasible path induced by the policy function (5.6). We compare this path with a candidate of optimal path $(x(t), c(t))$ starting from the same initial capital stock $x(0) = \tilde{x}(0)$. (A.34) leads to:

$$\int_0^\infty u(\tilde{c}(t))e^{-\rho t}dt - \int_0^\infty u(c(t))e^{-\rho t}dt \geq \lim_{t\to\infty} e^{-\rho t}(\tilde{V}^*(x(t)) - \tilde{V}^*(\tilde{x}(t))). \tag{A.35}$$

We show that the right-hand side of (A.35) is nonnegative. This is obvious if the utility function is bounded from below or both $x(t)$ and $\tilde{x}(t)$ converge to a positive number. In other words, we need to check the case that $\sigma \geq 1$ and either of $x(t)$ or $\tilde{x}(t)$ converges to 0. From Proposition 2.1, $(x(t), c(t))$ can be chosen in the class of the $x$-$c$ paths. Then since the Jacobian matrix of the $x$-$c$ system at $(0, 0)$ is

$$J = \begin{bmatrix} \alpha & -1 \\ 0 & (\alpha - \rho)/\sigma \end{bmatrix}, \tag{A.36}$$

the origin is a saddle point by (5.4). The stable eigenvector of (A.36) is given by $(1, \beta)$. For an $x$-$c$ path $(x(t), c(t))$ that converges to the origin with a sufficiently small initial value $x(0)$, we have:

$$x(t) = x(0)\exp\left[-\left(\frac{\rho-\alpha}{\sigma}\right)t\right] + o(x(t)),$$

where $\lim_{x\to 0} o(x) = 0$. Then we have:

$$\lim_{t\to\infty} e^{-\rho t}\tilde{V}^*(x(t)) = \begin{cases} \lim_{t\to\infty} \frac{\beta^{-\sigma}x(t)^{1-\sigma}e^{-\rho t}}{1-\sigma} = \lim_{t\to\infty} \frac{\beta^{-\sigma}x(0)^{1-\sigma}e^{-\beta t}}{1-\sigma} = 0 \text{ for } \sigma \neq 1 \\ \lim_{t\to\infty} e^{-\rho t}\left[\left(\ln \rho x(0)e^{(\alpha-\beta)t}\right)/\rho + (\alpha-\beta)/\rho^2\right] = 0 \text{ for } \sigma = 1 \end{cases}. \tag{A.37}$$

Therefore, if an $x$-$c$ path $(x(t), c(t))$ converges to the origin, then $\lim_{t\to\infty} e^{-\rho t}\tilde{V}^*(x(t)) = 0$. As $(\tilde{x}(t), \tilde{c}(t))$ is also an $x$-$c$ path, if it converges to the origin, we have $\lim_{t\to\infty} e^{-rt}\tilde{V}^*(\tilde{x}(t)) = 0$. Therefore the right-hand side of (A.35) is 0, and the proof completes. ■

## A.9 Proof of Lemma 5.2

Using the gain function (2.11), we have:

$$V_L(c^M) - \int_0^\infty f(x^s)e^{-\rho t}dt = \int_0^{T^*(c^M)} \gamma(x^M(t))e^{-\rho t}dt - \int_0^\infty \gamma(x^s)e^{-\rho t}dt. \tag{A.38}$$

It is easily verified that $T^*(c^M)$ and $V_L(c^M)$ are continuous. Given $\rho > \hat{\rho}$:

$$0 > \gamma(x^M(t)) \geq \gamma(x_s) \text{ for all } t \in [0, T^*(c^M)), \tag{A.39}$$

where $\check{x}_s$ is the lower stationary capital stock. As $\lim_{c^M \to \infty} T^*(c^M) = 0$, the following holds:

$$0 > \int_0^{T^*(c^M)} \gamma(x^M(t))e^{-\rho t}dt \geq \int_0^{T^*(c^M)} \gamma(x_s)e^{-\rho t}dt \to 0 \text{ as } c^M \to \infty. \tag{A.40}$$

Therefore:

$$\int_0^{T^*(c^M)} \gamma(x^M(t))e^{-\rho t}dt \to 0 \text{ as } c^M \to \infty. \tag{A.41}$$

On the other hand, as $\rho > \hat{\rho}$, $\gamma(x^s) < 0$, and thus $\int_0^\infty \gamma(x^s)e^{-\rho t}dt < 0$. Therefore, with a sufficiently large $c^M$, (A.38) is positive. That is, (5.12) holds. ■

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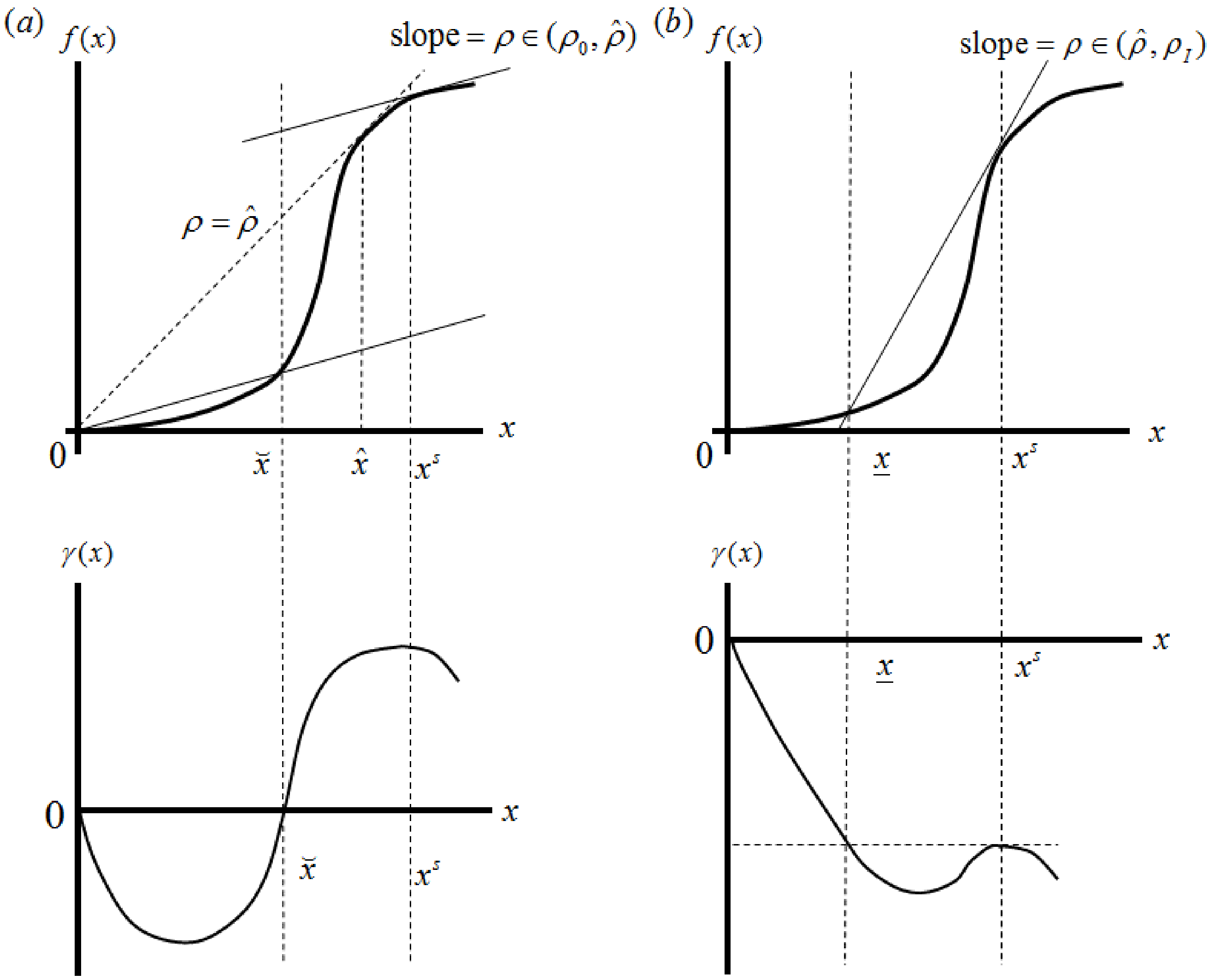


Figure 1. Production function and gain function

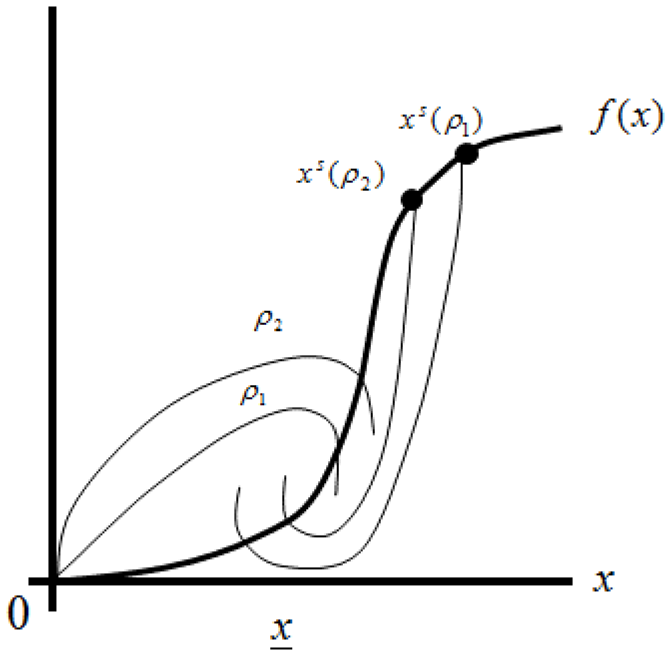


Figure 2. Ascending and descending paths when $\rho = \rho_1, \rho_2$ $(\rho_1 < \rho_2)$

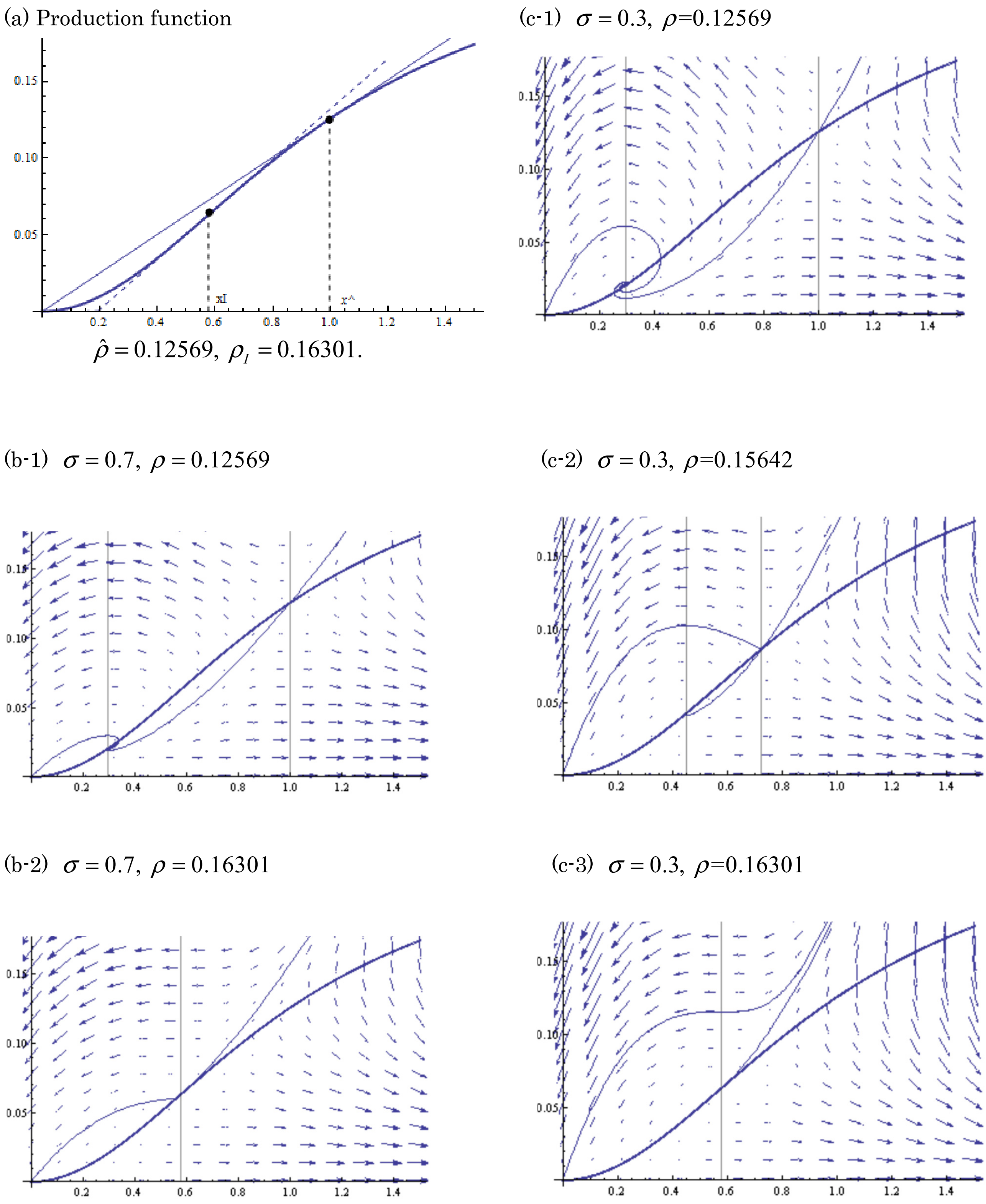


Figure 3. Numerical simulations.

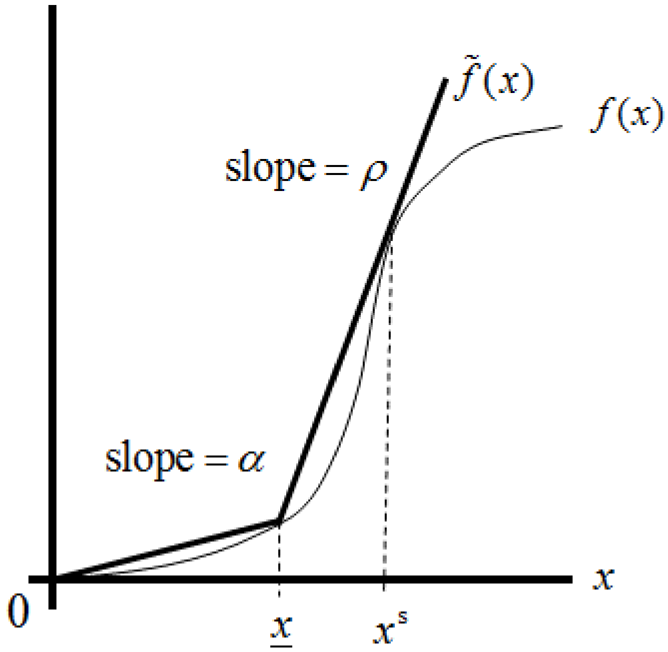


Figure 4. Piecewise linear production function