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in a continuous time nonconcave Ramsey model**

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# Existence of an optimal path in a continuous-time nonconcave Ramsey model

## **Abstract:**

We show an existence theorem for a continuous-time nonconcave Ramsey model. In existing existence theorems, a bounded condition is required to ensure the compactness of the set of feasible control paths. Although our existence theorem is for a specific Ramsey model, it does not require such a bounded condition. The continuous-time Ramsey model is extensively used. In many cases, the analysis has been conducted without explicit reference to the conditions for the existence of an optimal path or simply by assuming its existence. Our result provides validity to such analyses.

Keywords: Continuous-time nonconcave Ramsey model, existence of an optimal path

JEL Codes: C61, C62, O41

# 1 Introduction

The Ramsey model, an infinite time-horizon optimal control model with discounting, is one of the models in economics most extensively used to analyze intertemporal decision-making and the dynamics of economic systems. The existence of a solution is the keystone of the analysis. For the discrete-time model, the existence theorem is well established and a solution exists under economically natural assumptions. See, for example, McKenzie (1986), Stokey and Lucas (1988, Ch. 3) and Le Van (2006). In contrast, for the continuous-time model, existence has been proved with a bounded condition on feasible control (consumption and/or investment) paths, such as Magill's (1981, p.687) Assumption 3, Chichilnisky's (1981, p.55) conditions (i), (v) and (vi), Romer's (1986, p.899) condition (ii) in Theorem, and d'Albis, Gourdel and Le Van's (2008, p.323) assumption A3. These conditions are imposed to ensure the compactness of the set of feasible paths.<sup>1</sup> In contrast to the discrete-time model, there is no natural upper bound to consumption and/or investment. Because a bounded condition is assumed for a technical reason, usually the continuous-time Ramsey model has been used without explicit reference to it or simply by assuming the existence of an optimal path.<sup>2</sup>

When the model is a concave model, we can rely on Mangasarian's (Mangasarian, 1966) or Arrow's sufficiency theorems (Arrow and Kurz, 1970). In this case, if we find a

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<sup>1</sup>See Ekeland and Teman (1999, Chapter 8, Theorem 1.3) for the conditions used to ensure the compactness (weak compactness in an  $L^1$  space). For a nonconcave model, the nonexistence of solutions may emerge due to chattering. Clark (1976, Chapter 5) shows an example. Romer (1986) provides an explanation of the cause of chattering as well as of other causes of the nonexistence of solutions.

<sup>2</sup>There is an existence theorem that does not require a restriction to the feasible control set (Fleming and Soner, 2006, Theorem I.11.2). However, it is for a specific model that is rarely used in economics. In addition, in common with many control-theory models, it assumes Lipschitz continuity of the integrand of the objective functional and other functions, which excludes, for example, the use of the logarithmic utility function and the Cobb–Douglas production function.

feasible path that satisfies the sufficient conditions, it simultaneously proves the existence of an optimal path. However, in a nonconcave problem, these sufficiency theorems are not available.<sup>3</sup>

This paper provides an existence theorem for a continuous-time nonconcave model. The merit of our result is that there is no restriction to the set of feasible paths except for the nonnegativity condition. In contrast to the above papers, which seek a general theorem, we consider a simple nonconcave Ramsey model that was first analyzed by Skiba (1976). The model is a one-state variable, optimal control model with a concave utility function and a convex–concave production function. It is known that for a certain range of discount rates, there is a threshold capital stock such that any optimal capital path from a stock level below (above) this threshold converges to zero (an interior steady state). This property is interpreted as history dependence and polarization in economic phenomena and the model has been applied in a broad range of fields, including economic development, firm dynamics, public policy, international trade, and environmental and resource economics. See Akao, Kamihigashi and Nishimura (2019a) for a review of the literature. We use this model because, apart from its popularity, it exhibits all possible optimal paths of a one-state variable economic growth model with a standard utility function. Our result and the strategy to prove the existence of an optimal path can be applied beyond the specific model used here.

The rest of the paper is organized as follows. The next section describes the model and assumptions. This section also shows the basic properties of the optimal value function.

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<sup>3</sup>Although there are sufficiency theorems for a nonconcave optimal control problem (see Sorger, 1989), there are few applications. One exception is Akao, Kamihigashi and Nishimura (2019b). See also Footnote 4 of this paper.

The third section proves the existence of an optimal path.

## 2 Model, assumptions and some basic results

Consider a continuous-time optimal growth model:

$$V^*(x_0) := \sup_{c(t)} \int_0^\infty u(c(t)) e^{-\rho t} dt \quad (2.1)$$

subject to  $\dot{x}(t) = f(x(t)) - c(t)$ ,  $c(t) \geq 0$ ,  $x(t) \in X$ ,  $x(0) = x_0 \in X$  given,

where  $c(t)$  is the consumption path,  $x(t)$  is the capital path,  $x_0$  is the initial capital stock and  $\rho > 0$  is the discount rate. The state space  $X \subset \mathbb{R}_+$  is the closed interval with  $\min X = 0$  and can be  $X = \mathbb{R}_+$ .

A path  $(x(t), c(t))$  is called a feasible path starting from  $x_0$  if it satisfies the constraint in (2.1) and if  $x(t)$  is a unique solution of the state equation with the initial value  $x_0$ . Then, a feasible path  $(x^*(t), c^*(t))$  starting from  $x_0$  is optimal if there is no feasible path  $(x(t), c(t))$  starting from  $x_0$  that satisfies:

$$\int_0^\infty [u(c(t)) - u(c^*(t))] e^{-\rho t} dt > 0.$$

We make the following assumptions:

**Assumption 1:** The utility function  $u : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$  is twice continuously differentiable on  $(0, \infty)$ , and satisfies  $u'(c) > 0$ ,  $u''(c) < 0$  and  $\lim_{c \rightarrow 0} u'(c) = \infty$ .

**Assumption 2:** The production function  $f : \mathbb{R}_+ \mapsto \mathbb{R}$  is a twice continuously differentiable function with the following properties: (a)  $f(0) = 0$ , (b) there is an inflection point  $x_I$  such that  $f''(x) \geq 0$  for  $x \leq x_I$ , (c)  $\rho_0 := \lim_{x \rightarrow 0} f'(x) > 0$ , (d)  $\lim_{x \rightarrow 0} f''(x)$  exists and (e)  $\lim_{x \rightarrow \infty} f'(x) \leq 0$ .

**Assumption 3:** The initial value of capital stock is positive:  $x_0 > 0$ .

**Remarks:**

1. The model is nonconcave owing to Assumption 2 (b). The production function is strictly convex on  $[0, x_I]$  and strictly concave on  $[x_I, \infty)$ , and is called a convex-concave production function.
2. Assumption 3 excludes a trivial case  $x_0 = 0$ , where  $(x(t), c(t)) = (0, 0)$  is the only feasible path.

The problem (2.1) is well defined under Assumptions 1 and 2. That is:

**Proposition 2.1**  $V^*(x_0) < \infty$ .

**Proof.** See Appendix A.1. ■

We can also show the following:

**Proposition 2.2**  $V^* : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is (i) nondecreasing and (ii) continuous at  $x$ , such that  $f(x) > 0$ .

**Proof.** See Appendix A.2. ■

**Remark:** As seen in the proof,  $V^*$  is left continuous at  $x$ , such that  $x > 0$  and  $f(x) = 0$ .

We define the continuous-time version of the gain function (Kamihigashi and Roy, 2006, 2007) by:

$$\gamma(x) := f(x) - \rho x. \quad (2.2)$$

Using it, we show that we may restrict the state space to a compact set  $[0, \bar{x}]$ , where  $\bar{x}$  is chosen to satisfy:

$$f'(\bar{x}) < \min\{\rho, \rho_0\}. \quad (2.3)$$

**Proposition 2.3** *Consider a feasible path  $(x(t), c(t))$  starting from  $\bar{x}$ , such that  $x(t) > \bar{x}$  for  $t \in (0, T)$ , where  $T \in \mathbb{R}_{++} \cup \{\infty\}$  is the first exit time from  $(\bar{x}, \infty)$ . Let  $(x'(t), c'(t))$  be a feasible path such that:*

$$(x'(t), c'(t)) = \begin{cases} (\bar{x}, f(\bar{x})) & \text{for } t \in [0, T) \\ (x(t), c(t)) & \text{for } t \geq T \end{cases}.$$

*Then,  $(x(t), c(t))$  is dominated by  $(x'(t), c'(t))$ .*

**Proof.** See Appendix A.3. ■

From this proposition, an optimal path starting from  $x \in [0, \bar{x}]$  stays in  $[0, \bar{x}]$ . Hereafter, we consider the problem on the state space:

$$X = [0, \bar{x}], \quad (2.4)$$

where  $\bar{x}$  satisfies (2.3) and:

$$f(\bar{x}) \geq 0. \quad (2.5)$$



(2.5) ensures the continuity of the value function on  $(0, \bar{x}]$ .

**Remark:** If  $\lim_{x \rightarrow \infty} f'(x) = 0$ , as assumed in a neoclassical growth model, we can choose an arbitrarily large  $\bar{x}$ . If  $\lim_{x \rightarrow \infty} f'(x) < 0$ , as in a bioeconomic model, we can choose  $\bar{x}$  to satisfy  $f(\bar{x}) = 0$ . Then, if the initial stock is greater than  $\bar{x}$ , the stock moves in  $[0, \bar{x}]$  in a finite time with any consumption path because  $f'(x) < 0$  for  $x > \bar{x}$ . In both cases, the restriction of the state space by (2.3) and (2.5) is innocuous.

### 3 Existence theorem

#### 3.1 Modified model

The strategy for the proof of existence is as follows. Modify the problem (2.1) by imposing an upper bound of consumption. For the problem, we can apply d'Albis, Gourdel and Le Van's (2008) existence theorem and an optimal path exists. Then, we show that the optimal path is interior when the upper bound is sufficiently high and it is invariant to an upward shift of the upper bound. We take the limit of the upper bound and see that the optimality of the path is preserved, which completes the proof.

Choose  $\bar{c}$  to satisfy that  $\bar{c} > \max\{f(x) | x \in [0, \bar{x}]\}$ . We modify the problem (2.1) by imposing the maximum consumption  $\bar{c}$ :

$$V^*(x_0; \bar{c}) := \sup_{c(t)} \int_0^{\infty} u(c(t)) e^{-\rho t} dt \tag{3.1}$$

subject to  $\dot{x}(t) = f(x(t)) - c(t)$ ,  $c(t) \in [0, \bar{c}]$ ,  $x(t) \in [0, \bar{x}]$ ,  $x(0) = x_0 \in (0, \bar{x}]$ .

For this problem, we have the following proposition.

**Proposition 3.1** (i) For any  $x_0 \in [0, \bar{x}]$ , an optimal path starting from  $x_0$  exists. (ii)  $V^*(x_0; \bar{c})$  as a function of  $x_0$  is continuous and nonincreasing.

**Proof.** (i) As in the proof of Proposition 2.1, there are  $a \geq 0$  and  $b \in (0, \min\{\rho, \max f(x)/x\})$ , such that  $f(x) \leq a + bx$  for  $x \geq 0$ . Therefore, for a feasible path,  $x(t) \leq (a/b)e^{bt}$ . It is easily verified that the other conditions in d’Albis, Gourdel and Le Van (2008) are satisfied. Then, their Theorem 1 ensures the existence of an optimal path. (ii) The properties follow from the proof of Proposition 2.2, which is not affected by the imposition of an upper bound  $\bar{c}$ . ■

**Remark:** Romer’s (1986) existence theorem is also applicable. The felicity is written as  $v(x, \dot{x}) := u(f(x) - \dot{x})$ . Note that  $v$  is concave in  $\dot{x}$  and  $\dot{x}$  is bounded:  $|\dot{x}(t)| \leq \bar{c}$ .

## 3.2 Properties of an optimal path

If an optimal path is interior, i.e.,  $(x^*(t), c^*(t)) \in (0, \bar{x}) \times (0, \bar{c})$  for all  $t \geq 0$ , then it is a solution of the so-called canonical system of Hamiltonian differential equations:

$$\dot{x}^*(t) = f(x^*(t)) - u'^{-1}(\lambda(t)), \quad (3.2a)$$

$$\dot{\lambda}(t) = (\rho - f'(x^*(t))) \lambda(t), \quad (3.2b)$$

where  $\lambda(t)$  is the costate variable and  $u'^{-1}$  is the inverse function of  $u'$ . Let:

$$\sigma(c) := -\frac{cu''(c)}{u'(c)}. \quad (3.3)$$

The following system of differential equations is equivalent to the canonical system (3.2):

$$\dot{x}(t) = f(x(t)) - c(t), \quad (3.4a)$$

$$\dot{c}(t) = \frac{c(t)}{\sigma(c(t))} [f'(x(t)) - \rho]. \quad (3.4b)$$

We refer to it as the  $x$ - $c$  system and call a solution of the system an  $x$ - $c$  path. We use  $(x^s, f(x^s))$  to denote the steady state of the  $x$ - $c$  system that satisfies  $f'(x^s) = \rho$  and  $f''(x^s) < 0$ .  $(x^s, f(x^s))$  exists if  $\rho < \rho_I$ .<sup>4</sup>

An optimal path satisfies the following properties:

**Proposition 3.2** (i)  $(x^*(t), c^*(t)) > (0, 0)$  for all  $t \geq 0$ . (ii) If  $x^*(t)$  is not constant, it monotonically converges to 0 or  $x^s$ . (iii) If the optimal path is not unique, there are two optimal paths. One converges to the origin and the other converges to  $(x^s, f(x^s))$ . The initial stock under which two optimal paths exist is unique and lies in  $(0, x^s)$ .

**Proof.** See Appendix A.4. ■

The monotonicity above implies that if  $\dot{x}^*(t) < (>)0$  for some  $t$ , then  $\dot{x}^*(t) < (>)0$  for all  $t$ . We call this optimal capital path decreasing (increasing). If an optimal path contains  $c^*(t) = \bar{c}$ , we call it a bounded optimal path. The monotonicity above implies that a bounded optimal capital path is decreasing.

Point (iii) in the above proposition implies the following corollary.

**Corollary 3.1** *If an optimal capital path is decreasing (increasing), then there is no other decreasing (increasing) optimal capital path.*

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<sup>4</sup> $(x^s, f(x^s))$  may not be an optimal steady state. There is a  $\rho_H \in (\max f(x)/x, \rho_I)$  such that the steady state is not optimal if  $\rho > \rho_H$ . See Akao, Kamihigashi and Nishimura (2019a, Corollary 4.1).

To show a further result concerning a bounded optimal path, we define the Lagrangian and the maximized Hamiltonian for the problem (3.1):

$$L(c, x, \lambda, \mu) := u(c) + \lambda(f(x) - c) + \mu(\bar{c} - c), \quad (3.5)$$

and:

$$H^*(x, \lambda) := \max_{c \in [0, \bar{c}]} \{u(c) + \lambda(f(x) - c)\}. \quad (3.6)$$

From Pontryagin's maximum principle, for an optimal path  $(x^*(t), c^*(t))$ , there is a costate variable  $\lambda(t)$  and a Lagrange multiplier  $\mu(t)$ , and the following equations are satisfied:

$$u'(c^*(t)) = \lambda(t) + \mu(t), \quad \mu(t) \geq 0, \quad \mu(t)(\bar{c} - c^*(t)) = 0, \quad (3.7a)$$

$$\dot{\lambda}(t) = \lambda(t)(\rho - f(x^*(t))). \quad (3.7b)$$

From (3.7), we have:

**Lemma 3.1** (i) If  $c^*(t) < \bar{c}$  for  $t \in (t' - \varepsilon, t' + \varepsilon)$  with  $\varepsilon > 0$ ,  $c^*(t)$  is continuous at  $t'$ . (ii) If  $c^*(t) < \bar{c}$  for  $t \in (t' - \varepsilon, t')$  and  $c^*(t) = \bar{c}$  for  $t \in [t', t' + \varepsilon)$  with  $\varepsilon > 0$ ,  $c^*(t)$  is continuous at  $t'$ . (iii) If  $c^*(t) = \bar{c}$  for  $t \in (t' - \varepsilon, t')$  and  $c^*(t) < \bar{c}$  for  $t \in [t', t' + \varepsilon)$  with  $\varepsilon > 0$ ,  $c^*(t)$  is continuous at  $t'$ .

**Proof.** (i) On  $(t' - \varepsilon, t' + \varepsilon)$ ,  $c^*(t)$  satisfies  $u'(c^*(t)) = \lambda(t)$ . Then, the continuity follows from the fact that  $\lambda(t)$  is continuous and  $u'(c)$  is continuous. (ii) Assume that  $\lim_{t \nearrow t'} c^*(t) < \bar{c}$ . Then,  $\lim_{t \nearrow t'} \lambda(t) > u'(\bar{c}) = \lambda(t') + \mu(t') \geq \lambda(t')$ , which contradicts the continuity of  $\lambda(t)$ . (iii) Similarly, assume that  $\lim_{t \searrow t'} c^*(t) < \bar{c}$ . Then,  $\lim_{t \searrow t'} \lambda(t) >$

$u'(\bar{c}) = \lambda(t') + \mu(t') \geq \lambda(t')$ , which contradicts the continuity of  $\lambda(t)$ . ■

Using this continuity result, we have the following proposition.

**Proposition 3.3** *If a sufficiently large  $\bar{c}$  is chosen,  $(x^*(t), c^*(t))$  satisfies  $c^*(t) \in (0, \bar{c})$  for all  $t \geq 0$ , i.e., an optimal path is interior. Furthermore, the optimality of the path is preserved in the case of an upward shift of the upper bound from  $\bar{c}$ .*

**Proof.** From Proposition 3.2 (ii), an optimal path eventually lies on the stable manifold of the  $x$ - $c$  system at the origin or  $(x^s, f(x^s))$ . As seen from the vector fields of the system, owing to the compact state space, there is a maximum consumption level of these manifolds. Take a  $\bar{c}$  larger than this level. Then, Lemma 3.1 ensures that an optimal path is interior.

Next, consider the optimality preserving property. We use  $(x^*(t; x_0, c), c^*(t; x_0, c))$  to denote an optimal path starting from  $x_0$  with the upper bound  $c$ . Let  $\bar{c}' > \bar{c}$  where  $\bar{c}$  is taken, as above. Assume that a new optimal path  $(x^*(t; x_0, \bar{c}'), c^*(t; x_0, \bar{c}'))$  appears. Because it is new, there is a consumption level:

$$c^*(t; x_0, \bar{c}') \in (\bar{c}, \bar{c}') \tag{3.8}$$

and the optimal capital path is decreasing. Then, there is a time  $t'$  such that the optimal path is interior for  $t > t'$ , and  $(x^*(t; x_0, \bar{c}'), c^*(t; x_0, \bar{c}')) = (x^*(t; x_0, \bar{c}), c^*(t; x_0, \bar{c}))$  for  $t > t'$ . This implies that they lie in the same manifold and, from the definition of  $\bar{c}$ ,  $c^*(t; x_0, \bar{c}') < \bar{c}$ , which contradicts (3.8). ■

### 3.3 Existence theorem

Now, we turn to the original model (2.1).

**Lemma 3.2** *Let  $\bar{c} > \max\{f(x)|x \in [0, \bar{x}]\}$ . Let  $\{\bar{c}_i\}_{i=1}^{\infty}$  be a sequence such that  $\bar{c} \leq \bar{c}_i < \bar{c}_{i+1}$  and  $\lim_{i \rightarrow \infty} \bar{c}_i = \infty$ . Given a feasible path  $(x(t), c(t))$ , such that  $x(t) \in [0, \bar{x}]$  for all  $t \geq 0$ , we define  $\bar{c}_i(t)$  by:*

$$\bar{c}_i(t) = \begin{cases} \bar{c}_i & \text{if } c(t) > \bar{c}_i \\ c(t) & \text{if } c(t) \leq \bar{c}_i \end{cases}.$$

*Then,  $\bar{c}_i(t)$  is a feasible consumption path and:*

$$\lim_{i \rightarrow \infty} \int_0^{\infty} [u(c(t)) - u(\bar{c}_i(t))] e^{-\rho t} dt = 0. \quad (3.9)$$

**Proof.** Fix the initial value  $x_0 \in (0, \bar{x}]$ . Let  $\phi(t; c)$  be the solution of  $\dot{x}(t) = f(x(t)) - c(t)$  with the initial value  $x_0$ . Similarly, let  $\phi(t; \bar{c}_i)$  be the solution with the same initial value when the control is  $\bar{c}_i(t)$ . For  $t \geq 0$ , such that  $\phi(t; c) = \phi(t; \bar{c}_i)$ ,  $\dot{\phi}(t; c) \leq \dot{\phi}(t; \bar{c}_i)$ , as  $c(t) \geq \bar{c}_i(t)$ . This implies that  $0 \leq \phi(t; c) \leq \phi(t; \bar{c}_i)$ : the nonnegativity condition on the state variable is satisfied with  $\bar{c}_i$ . Note that  $\sup_{t \geq 0} \phi(t; \bar{c}_i) \leq \bar{x}$  because  $\bar{c} > f(\bar{x})$ . The uniqueness of the solution  $\phi(t; \bar{c}_i)$  follows from the fact that both  $\dot{x}(t) = f(x(t)) - c(t)$  and  $\dot{x}(t) = f(x(t)) - \bar{c}_i$  have unique local solutions. From these,  $\bar{c}_i(t)$  is a feasible consumption path. (3.9) follows from the monotone convergence theorem. ■

Below is our result on the existence of an optimal path.

**Theorem 3.1** *The problem (2.1) has an optimal path. It is an interior optimal path, a solution of the  $x$ - $c$  system (3.4).*

**Proof.** Let  $(x^*(t), c^*(t))$  be an optimal path to the modified problem (3.1), with a sufficiently high  $\bar{c}$ . From Proposition 3.3,  $(x^*(t), c^*(t))$  is a solution of the  $x$ - $c$  system (3.4). Let  $(x(t), c(t))$  be a feasible path starting from the same initial capital stock, i.e.,  $x(0) = x^*(0)$ . Let  $c(t; \bar{c})$  be the path defined by:

$$c(t; \bar{c}) = \begin{cases} \bar{c} & \text{if } c(t) > \bar{c} \\ c(t) & \text{if } c(t) \leq \bar{c} \end{cases}.$$

As shown in Lemma 3.3,  $c(t; \bar{c})$  is feasible. Then:

$$\begin{aligned} & \int_0^\infty [u(c(t)) - u(c^*(t))] e^{-\rho t} dt \\ &= \lim_{\bar{c} \rightarrow \infty} \int_0^\infty [u(c(t)) - u(c(t; \bar{c})) + u(c(t; \bar{c})) - u(c^*(t))] e^{-\rho t} dt \\ &= \lim_{\bar{c} \rightarrow \infty} \int_0^\infty [u(c(t)) - u(c(t; \bar{c}))] e^{-\rho t} dt + \lim_{\bar{c} \rightarrow \infty} \int_0^\infty [u(c(t; \bar{c})) - u(c^*(t))] e^{-\rho t} dt \\ &\leq 0, \end{aligned}$$

from (3.9) in Lemma 3.3 and the fact that  $(x^*(t), c^*(t))$  is an optimal path to the modified problem (3.1). Note that the sequence in the second line and the sequence in the first term in the third line converge and, thus, the sequence in the second term in the third line also converges. ■

**Remarks:**

1. We may relax Assumption 2 (e) with  $\lim_{x \rightarrow \infty} f'(x) \leq g$  with some  $g > 0$ . In this case, Theorem 3.1 is modified, as an optimal path exists when  $\rho > g$ .

2. Although we consider a convex–concave production function, the theorem is valid for a concave Ramsey model.



# A Appendix: Some proofs

## A.1 Proposition 2.1

Take  $b \in (0, \min\{\rho, \max f(x)/x\})$  and let  $x_b$  satisfy that  $f'(x_b) = b$  and  $f''(x_b) < 0$ . Choose  $a$  to satisfy  $a \geq f(x_b) - bx_b$ . With  $a, b$ :

$$f(x) \leq a + bx$$

holds for all  $x$ . Let  $(x(t), c(t))$  be a feasible path and let  $\dot{x}(t) = f(x(t)) - c(t)$ . Then, we have:

$$\begin{aligned} \int_0^\infty u(c(t))e^{-\rho t} dt &= \frac{1}{\rho} \int_0^\infty u[f(x(t)) - \dot{x}(t)] \rho e^{-\rho t} dt \\ &\leq \frac{1}{\rho} \int_0^\infty u[a + bx(t) - \dot{x}(t)] \rho e^{-\rho t} dt \\ &= \frac{1}{\rho} \int_0^\infty u \left[ a - \frac{x(t)e^{-bt}}{dt} e^{bt} \right] \rho e^{-\rho t} dt \\ &\leq \frac{1}{\rho} u \left( \int_0^\infty \left[ a - \frac{dx(t)e^{-bt}}{dt} e^{bt} \right] \rho e^{-\rho t} dt \right) \\ &= \frac{1}{\rho} u \left( a - [\rho x(t)e^{-\rho t}]_0^\infty - \rho \int_0^\infty (\rho - b) x(t)e^{-\rho t} dt \right) \\ &\leq \frac{1}{\rho} u \left( a + \rho \left( x(0) - \lim_{t \rightarrow \infty} x(t)e^{-\rho t} \right) \right) \\ &= \frac{1}{\rho} u(a + \rho x(0)). \end{aligned}$$

The fourth line uses Jensen's inequality. The last equality is obtained as follows. Because:

$$\dot{x}(t) = f(x(t)) - c(t) \leq f(x(t)) \leq a + bx(t),$$

$\dot{x}(t) - bx(t) \leq a$ . This implies that  $dx(t)e^{-bt}/dt \leq ae^{-bt}$ . Then, integrate both sides:

$$\int_0^T \frac{dx(t)e^{-bt}}{dt} dt = x(T)e^{-bT} - x(0) \leq \frac{a}{b} (1 - e^{-bT}).$$

Therefore, as  $T \rightarrow \infty$ :

$$x(T)e^{-\rho T} \leq \left( \frac{a}{b} (1 - e^{-bT}) + x(0) \right) e^{-(\rho-b)T} \rightarrow 0.$$

■

## A.2 Proposition 2.2

Let  $x, y \in X$  and  $x > y > 0$ . (i) Let  $c(t; y)$  be a feasible consumption path starting from  $y$ . Let  $\phi(t; z)$  be the solution of  $\dot{x}(t) = f(x(t)) - c(t; y)$ , with the initial value  $z$ . Assume that  $c(t; y)$  is not a feasible consumption path when the initial stock is  $x$ . This means that there is a time  $T \in (0, \infty)$  such that  $\phi(T; x) = 0$  and  $c(T; y) > 0$ . On the other hand,  $\phi(T; y) > 0$ . (Otherwise,  $c(t; y)$  is not feasible when  $y$  is the initial stock either.)

Therefore, we have:

$$\phi(T; y) > 0 = \phi(T; x).$$

This inequality with the assumption that  $x > y$  implies that there is  $T' \in (0, T)$  such that  $\phi(T'; x) = \phi(T'; y) > 0$ . However, this contradicts the uniqueness of a solution. Therefore,  $c(t; y)$  is a feasible path starting from  $x$ , and we have  $V^*(x) \geq V^*(y)$ . (ii) Let  $x, y$  satisfy

that  $f(x) > 0$  and  $f(y) > 0$ . Note that:

$$\min_{z \in [y, x]} f(z) = \min\{f(x), f(y)\} > 0$$

from Assumption 2. Choose a constant  $c \in (0, \min_{z \in [y, x]} f(z))$  and consider the solution of  $\dot{x}(t) = f(x(t)) - c$  with the initial value  $y$ . The solution monotonically reaches  $x$  at a finite time  $T(x, y; c)$ , where the function  $T$  satisfies:

$$\frac{\partial T(z, y; c)}{\partial z} = \frac{1}{f(z) - c}. \quad (\text{A.1})$$

Let:

$$K := \max_{z \in [y, x]} \frac{1}{f(z) - c}.$$

From the dynamic programming principle:

$$V^*(y) \geq \int_0^{T(x, y; c)} u(c) e^{-\rho t} dt + e^{-\rho T(x, y; c)} V^*(x).$$

Then, we have:

$$\begin{aligned}
|V^*(x) - V^*(y)| &= V^*(x) - V^*(y) \\
&\leq - \int_0^{T(x,y;c)} u(c) e^{-\rho t} dt + (1 - e^{-\rho T(x,y;c)}) V^*(x) \\
&= - \int_y^x \frac{u(c) e^{-\rho T(z,y;c)}}{f(z) - c} dz + \left( 1 - \exp \left[ -\rho \int_y^x \frac{dz}{f(z) - c} \right] \right) V^*(x) \\
&\leq |u(c)| K(x - y) + (1 - \exp[-\rho K(x - y)]) V^*(x) \\
&\rightarrow 0 \text{ as } |x - y| \rightarrow 0.
\end{aligned}$$

■

### A.3 Proposition 2.3

Because  $f'(\bar{x}) < \rho_0$ ,  $\gamma'(x) < 0$  in  $[\bar{x}, \infty)$  and, thus,  $\gamma(\bar{x}) > \gamma(x)$  for all  $x > \bar{x}$ . Then, owing to the strict concavity of  $u$ , we have:

$$\begin{aligned}
\int_0^T u(c(t)) e^{-\rho t} dt &\leq \frac{1 - e^{-\rho T}}{\rho} u \left( \frac{\rho}{1 - e^{-\rho T}} \int_0^T c(t) e^{-\rho t} dt \right) \\
&= \frac{1 - e^{-\rho T}}{\rho} u \left[ \frac{\rho}{1 - e^{-\rho T}} \left( \int_0^T \gamma(x(t)) e^{-\rho t} - \frac{dx(t) e^{-\rho t}}{dt} dt \right) \right] \\
&< \frac{1 - e^{-\rho T}}{\rho} u \left[ \frac{\rho}{1 - e^{-\rho T}} \left( \int_0^T \gamma(x_0) e^{-\rho t} dt + x_0 (1 - e^{-\rho T}) \right) \right] \\
&= \frac{1 - e^{-\rho T}}{\rho} u(f(x_0)). \tag{A.2}
\end{aligned}$$

This shows that the path staying at  $\bar{x}$  during  $T$  periods dominates the original path. ■

## A.4 Proposition 3.2

(i)  $(x^*(t), c^*(t)) > (0, 0)$  for all  $t \geq 0$ .

If an optimal capital path  $x^*(t)$  satisfies  $x^*(t) > 0$  for all  $t \geq 0$ , then, from Michel's (1982) theorem, the costate variable of the system of Hamiltonian differential equations  $u' [c^*(t)]$  exists for all  $t \geq 0$ , which implies that  $c^*(t) > 0$  by  $\lim_{c \rightarrow 0} u'(c) = \infty$  in Assumption 1. Therefore, we consider the case in which there exists the finite extinction time  $T^* := \min\{t \in \mathbb{R}_+ | x^*(t) = 0\}$ . Because  $x^*(t) = 0$  implies that  $c^*(t) = 0$ , the case of  $u(0) = -\infty$  is ruled out. Assume that  $u(c)$  is bounded from below and standardize it as  $u(0) = 0$ . Then, there is a tuple  $(c^*(t), x^*(t), T^*)$  that is a solution to a free final time problem with the constraint  $x(T) \geq 0$ :

$$\max_{c(t) \in [0, \bar{c}], T > 0} \int_0^T u(c(t)) e^{-\rho t} dt$$

$$\text{subject to } \dot{x}(t) = f(x(t)) - c(t), \quad x(t) \in [0, \bar{x}], \quad x(T) \geq 0, \quad x(0) \in (0, \bar{x}].$$

From Seierstad and Sydsæter (1987, Chapter 2, Theorem 11), at the extinction time  $T^*$ , there exists  $c^*(T^*) > 0$  such that:

$$u(c^*(T^*)) + u' [c^*(T^*)] (-c^*(T^*)) = 0. \tag{A.3}$$

Note that  $u$  is strictly concave and that  $u(0) = 0$ . Then, we have:

$$\begin{aligned} & u(c^*(T^*)) - u'(c^*(T^*))c^*(T^*) \\ &= [u(c^*(T^*)) - u(0)] - u'(c^*(T^*)) [c^*(T^*) - 0] > 0, \end{aligned}$$

which contradicts (A.3). Therefore,  $x^*(t) > 0$  for all  $t \geq 0$ . ■

**(ii) A nonconstant optimal capital path monotonically converges to 0 or  $x^s$ .**

First, we prove the monotonicity. Assume that there is an optimal path for which the capital path is not monotonic. We assume that  $t_1 \geq 0$  and  $t_2 > t_1$ , such that  $x^*(t_1) = x^*(t_2)$  and  $\dot{x}^*(t_1)\dot{x}^*(t_2) < 0$ . Let  $\bar{t} = \arg \max\{x^*(t) | t \in [t_1, t_2]\}$ . Consider the problem (3.1) with the initial stock  $x^*(\bar{t})$ . Because of the autonomous nature of the problem, the following period  $t_2 - t_1$  capital path should be optimal:

$$x(t) = x^* \left( t + \bar{t} - \left\lfloor \frac{t + \bar{t} - t_1}{t_2 - t_1} \right\rfloor (t_2 - t_1) \right).$$

( $\lfloor \cdot \rfloor$  is the floor function.) However, this path is dominated by the path remaining at  $x^*(\bar{t})$  from Proposition 2.3.

As an optimal capital path is monotonic and the state space is a closed interval  $[0, \bar{x}]$ , a nonconstant optimal capital path converges to 0,  $\bar{x}$  or the capital stock of a steady state of the  $x$ - $c$  system, if it exists. It is easily verified that  $(x^s, f(x^s))$  is a saddle, whereas another possible steady state  $(x_s, f(x_s))$ , such that  $f'(x_s) = \rho$  and  $f''(x_s) > 0$ , is unstable. Therefore, if a nonconstant optimal path converges to an interior point, the convergent

point must be  $x^s$ . Finally, note that  $\gamma'(x) < 0$  in a neighborhood of  $\bar{x}$ . Then, from Proposition 2.3, there is no optimal capital path that converges to  $\bar{x}$ .

**(iii) If an optimal path is not unique, there are two optimal paths. One converges to the origin and the other converges to  $(x^s, f(x^s))$ . The initial stock at which two optimal paths exist is unique and lies in  $(0, x^s)$ .**

Let  $(x_1^*(t), c_1^*(t))$  and  $(x_2^*(t), c_2^*(t))$  be two optimal paths starting from the same initial stock  $x_0$ . Assume that they are decreasing, i.e.,  $c_1^*(t) > f(x_1^*(t))$  and  $c_2^*(t) > f(x_2^*(t))$ . Let  $c_1^*(0) > c_2^*(0)$ . Michel (1982, Theorem) proves that an optimal path satisfies the terminal condition:

$$\lim_{t \rightarrow \infty} H^*(x_i^*(t), \lambda(t))e^{-\rho t} = 0, \quad i = 1, 2, \quad (\text{A.4})$$

where  $H^*$  is the maximized Hamiltonian defined in (3.6). Then, as shown in Davidson and Harris (1981, Appendix):

$$\int_0^\infty u(c_i^*(t))e^{-\rho t} dt = \rho^{-1} H^*(x_0, \lambda_i(0)), \quad i = 1, 2 \quad (\text{A.5})$$

holds. Note that  $H^*(x_0, \lambda)$  is strictly convex in  $\lambda$  with the minimum at  $\lambda = u' [f(x_0)]$ . In addition, note that:

$$u' [f(x_0)] < \lambda_1(0) \leq u'(c_1^*(0)) < c_2^*(0) = \lambda_2(0).$$

Therefore, we have a contradiction:

$$V^*(x_0) = \int_0^\infty u(c_1^*(t))e^{-\rho t} dt < \int_0^\infty u(c_2^*(t))e^{-\rho t} dt = V^*(x_0).$$

A similar argument is applied to show that there is no more than one increasing optimal path. If there are two optimal paths starting from  $x_0$ , one converges to the origin and the other converges to  $(x^s, f(x^s))$ . This implies that  $x_0 \in (0, x^s)$ . If there are two initial stocks  $x_0, x'_0$  from which two optimal paths exist, then we have another optimal path for which the capital path cyclically goes to  $x_0$  and comes back to  $x'_0$ . However, this path contradicts the monotonicity of an optimal capital path. ■



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