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Ken-Ichi Akao, Takashi Kamihigashi, & Kazuo Nishimura

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Ken-Ichi Akao

School of Social Sciences, Waseda University, Japan

akao@waseda.jp

Takashi Kamihigashi

Research Institute for Economics and Business Administration, Kobe University, Japan

tkamihig@rieb.kobe-u.ac.jp

Kazuo Nishimura

Institute of Economic Research, Kyoto University, Japan

nishimura@kier.kyoto-u.ac.jp

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Abstract: In a nonconcave economic dynamics model, an open question is the optimality of a steady state of the canonical system of Hamiltonian differential equations in the convex part of the production function. We demonstrate that it can be an optimal steady state.

Keywords: Economic dynamic model, Convex-concave production function, Optimal steady state

JEL codes: C61, D90, O41.

1 Introduction

We consider an economic dynamics model with a convex-concave production function. This type of model appears in a broad range of economics such as economic development (Azariadis and Drazen 1990; Askenazy and Le Van 1999), firm dynamics (Davidson and Harris 1981; Haunschmied et al. 2005) and environmental and resource management (Clark 1971; Dasgupta and Mäler 2003), where increasing returns to scale prevails in small stock levels due to, for example, the large amount of initial investment and the nonconvexity of nature.

The simplest form of such a model is given by:

$$\max_{c(t)\geq 0} \int_0^\infty u\left(c(t)\right) e^{-\rho t} dt \tag{1}$$

$$\text{subject to } \dot{x}(t) = f\left(x(t)\right) - c(t),\ x(t) \geq 0,\ x(0) = x \geq 0 \text{ given,}$$

where the utility function $u : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ is a strictly increasing and strictly concave C^1 function on $(0, \infty)$ with $\lim_{c\searrow 0} u'(c) = \infty$, the discount rate ρ is positive, and the production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a C^2 function with the following properties: (a) $f(0) = 0$, (b) there is an inflection point x_I such that $f''(x) \gtreqless 0$ for $x \lesseqgtr x_I$, (c) $\lim_{x\searrow 0} f'(x) \geq 0$, and (d) $\lim_{x\nearrow\infty} f'(x) < \rho$.

When we assume $\lim_{x\searrow 0} f'(x) < \rho < f'(x_I)$, there is x_s such that $x_s \in (0, x_I)$ and $f'(x_s) = \rho$. In a concave model, $f'(x_s) = \rho$ implies that x_s is an optimal steady state. However, it may not be true because the problem is not concave and thus the Arrow's sufficiency theorem for an optimal path is not applicable. In fact, there are sufficient conditions that x_s can not be an optimal steady state. See Dechert and Nishimura (1983), Askenazy and Le Van (1999) and Akao, Kamihigashi and Nishimura (2019).

The question whether x_s can be an optimal steady state is concerned with the critical capital stock,[1] a threshold such that any optimal capital path from a stock level below (above) the threshold converges to a lower (higher) steady state. The critical capital stock has an important implication to economic problems such as the poverty trap and the ruinous use of environmental assets. If x_s is an optimal steady state, then it is the critical capital stock. However if it is not, the identification on the location is difficult in general.

In this paper, we demonstrate that x_s can be an optimal steady state by applying Sorger's sufficiency theorem for an optimal path. The next section shows the Sorger's theorem. The third section exemplifies that x_s can be an optimal steady state under certain conditions.

2 Sorger's sufficiency theorem

In this section, we introduce Sorger's sufficiency theorem for an optimal path. The theorem below is tailored from his original one for a general class of economic dynamics models to our one state variable autonomous model. Because of this, we shall also include the proof of the theorem.

For problem (1), path $(x(t), c(t))$ is called feasible if $x(t)$ is absolutely continuous, $c(t)$ is measurable and they satisfy the constraint in (1). If a feasible path $(x^*(t), c^*(t))$ satisfies

$$\int_0^\infty [u(c(t)) - u(c^*(t))] e^{-\rho t} dt \leq 0$$

for any feasible path $(x(t), c(t))$ such that $x(0) = x^*(0)$, then $(x^*(t), c^*(t))$ is a solution of (1) and is called

[1] It is also known as Skiba point or Dechert-Nishimura-Skiba (DNS) point in the literature. Skiba (1978) indicated the existence and Dechert and Nishimura (1983) proved it.

optimal.

We define the Hamiltonian and the maximized Hamiltonian H^* associated with problem (1) by

$$H(c, x, q) \equiv u(c) + q(f(x) - c), \tag{2}$$

$$\text{and } H^*(x, q) \equiv \max_{c \geq 0} H(c, x, q), \tag{3}$$

where q is the costate variable. Also, we define the function

$$\tilde{H}(x; q(t), Q(t), y(t)) \equiv H^*(x, q(t) - Q(t)(y(t) - x)) + \frac{1}{2}(\dot{Q}(t) - \rho Q(t))x^2 \tag{4}$$

with given absolutely continuous functions $q(t)$, $Q(t)$ and $y(t)$. Note that at $x = y(t)$,

$$\frac{\partial \tilde{H}(y(t); q(t), Q(t), y(t))}{\partial x} = \frac{\partial H^*(y(t), q(t))}{\partial x} + \frac{\partial H^*(y(t), q(t))}{\partial q} Q(t) + (\dot{Q}(t) - \rho Q(t))y(t), \tag{5}$$

if the maximized Hamiltonian is differentiable at $(y(t), q(t))$.

Theorem 1 *(Sorger, 1989, Lemma 1 and Theorem 1) Let $(x^*(t), c^*(t))$ be a feasible path for problem (1). $(x^*(t), c^*(t))$ is an optimal path if there exist absolutely continuous functions $q : [0, \infty) \mapsto \mathbb{R}$ and $Q : [0, \infty) \mapsto \mathbb{R}$ such that the following conditions are satisfied:*

(a) For almost all $t \in [0, \infty)$ the maximum condition

$$H^*(x^*(t), q(t)) = H(x^*(t), c^*(t), q(t)) \tag{6}$$

holds.

(b) The pair (x^*, p^*) satisfies the canonical system of differential equations

$$\dot{x}^*(t) = H_q^*(x^*(t), q(t)) \tag{7a}$$

$$\dot{q}(t) = \rho q(t) - H_x^*(x^*(t), q(t)) \tag{7b}$$

almost everywhere on $[0, \infty)$.

(c) For almost all $t \in [0, \infty)$ *the function* $\tilde{H}(x; q(t), Q(t), x^*(t))$ *in (4) is concave with respect to* x.

(d) For all feasible path from the same initial state with $(x^*(t), c^*(t))$, *the transversality condition*

$$\lim_{T \to \infty} e^{-\rho t} \left[q(t)(x(t) - x^*(t)) + \frac{1}{2} Q(t)(x(t) - x^*(t))^2 \right] \geq 0 \tag{8}$$

is satisfied.

Proof.

$$\int_0^\infty [u(c(t)) - u(c^*(t))]\, e^{-\rho t} dt$$

$$= \int_0^\infty [H(x(t), c(t), q(t)) - H(x^*(t), c^*(t), q(t)) + q(t)\,(\dot{x}^*(t) - \dot{x}(t))]\, e^{-\rho t} dt$$

$$= \int_0^\infty \left[\begin{array}{r} H[x(t), c(t), q(t) - Q(t)(x^*(t) - x(t))] - H(x^*(t), c^*(t), q(t)) \\ +q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt$$

$$\leq \int_0^\infty \left[\begin{array}{r} H^*[x(t), q(t) - Q(t)(x^*(t) - x(t))] - H^*(x^*(t), q(t)) \\ +q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt \qquad \text{(by Condition (a))}$$

$$= \int_0^\infty \left[\begin{array}{l} \tilde{H}(x(t); q(t), Q(t), x^*(t)) - \tilde{H}(x^*(t); q(t), Q(t), x^*(t)) \\ -(1/2)(\dot{Q}(t) - \rho Q(t))[x(t)^2 - x^*(t)^2] + q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt$$

$$\leq \int_0^\infty \left[\begin{array}{l} [\partial \tilde{H}(x^*(t); q(t), Q(t), x^*(t))/\partial x](x(t) - x^*(t)) \\ -(1/2)(\dot{Q}(t) - \rho Q(t))[x(t)^2 - x^*(t)^2] + q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt$$

(by Condition (c))

$$= \int_0^\infty \left[\begin{array}{l} [H_x^*(x^*(t), q(t)) + H_q^*(x^*(t), q(t))Q(t) + (\dot{Q}(t) - \rho Q(t))x^*(t)](x(t) - x^*(t)) \\ -(1/2)(\dot{Q}(t) - \rho Q(t))[x(t)^2 - x^*(t)^2] + q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt$$

(by (5))

$$= \int_0^\infty \left[\begin{array}{l} [-\dot{q}(t) + \rho q(t) + \dot{x}^*(t)Q(t) + (\dot{Q}(t) - \rho Q(t))x^*(t)](x(t) - x^*(t)) \\ -(1/2)(\dot{Q}(t) - \rho Q(t))[x(t)^2 - x^*(t)^2] + q(t)\,(\dot{x}^*(t) - \dot{x}(t)) + Q(t)(x^*(t) - x(t))\dot{x}(t) \end{array} \right] e^{-\rho t} dt$$

(by Condition (b))

$$= \int_0^\infty \left[\begin{array}{l} -(\dot{q}(t) - \rho q(t))(x(t) - x^*(t)) - q(t)\,(\dot{x}(t) - \dot{x}^*(t)) \\ -(1/2)(\dot{Q}(t) - \rho Q(t))(x(t) - x^*(t))^2 - Q(t)(x(t) - x^*(t))\,(\dot{x}(t) - \dot{x}^*(t)) \end{array} \right] e^{-\rho t} dt$$

$$= \int_0^\infty \frac{d}{dt} \left[\left(-q(t)(x(t) - x^*(t)) - (1/2)Q(t)(x(t) - x^*(t))^2 \right) e^{-\rho t} \right] dt$$

$$= -\lim_{t \to \infty} \left(q(t)(x(t) - x^*(t)) + (1/2)Q(t)(x(t) - x^*(t))^2 \right) e^{-\rho t}$$

$$\leq 0. \qquad \text{(by Condition (d))}$$

■

3 Optimal steady state: Example

Assume that the production function in the problem (1) takes the form of:

$$f(x) = -\frac{1}{6}b_3x^3 + \frac{1}{2}b_2x^2 + b_1x, \tag{9}$$

where $b_i > 0$ $(i = 1, 2, 3)$. $f(x)$ is strictly convex on $[0, b_2/b_3]$ and strictly concave on $[b_2/b_3, \infty)$, i.e. $x_I = b_2/b_3$. Let

$$\rho_I \equiv f'(x_I) = b_1 + \frac{b_2^2}{2b_3}. \tag{10}$$

We assume $\rho \in (b_1, \rho_I)$. Then, since $b_1 = \lim_{x\searrow 0} f'(x)$ and $\rho_I = \max\{f'(x)|x > 0\}$, there are two capital stocks that satisfy $f'(x) = \rho$. We call the smaller one the lower stationary capital stock and the larger one the upper stationary capital stock, and denote them by, respectively, $x_s(\rho)$ and $x^s(\rho)$ as functions of ρ. They are given by

$$x_s(\rho) = \frac{1}{b_3}\left\{b_2 - \sqrt{b_2^2 - 2b_3(\rho - b_1)}\right\}, \tag{11}$$

$$x^s(\rho) = \frac{1}{b_3}\left\{b_2 + \sqrt{b_2^2 - 2b_3(\rho - b_1)}\right\}. \tag{12}$$

With the production function (9), we shall show that the conditions in Sorger's sufficiency theorem (Theorem 1 in this note) are satisfied at the lower stationary capital stock $x_s(\rho)$, if ρ is close to b_1 and if certain curvature conditions are satisfied. The curvature conditions are expressed with the notion of α-concavity. A function $g : \mathbb{R} \mapsto \mathbb{R}$ is α-concave if $g(x) + (\alpha/2)\,|x|^2$ is concave. We assume that the utility function in the problem (1) is a-concave with large positive a in the relevant sub-domain. Applying the definition of α-concavity, we can say that the production function (9) is $-b_2$-concave. Note that b_2 expresses the degree of convexity. As we show in Proposition 1 below, in order for $x_s(\rho)$ to be an optimal steady state, the degree of concavity a in the utility function should increase as the degree of convexity

b_2 in the production function becomes large.

Proposition 1 *If the production function (9) satisfies*

$$b_2 < \frac{\sqrt{b_1 b_3}}{2}, \tag{13}$$

then there exist a utility function u *and the discount rate* $\bar{\rho}$ *such that* $x_s(\rho)$ *is an optimal steady state for* $\rho \in (b_1, \bar{\rho})$. *The utility function satisfies*

$$u'(c) = -a(c - f(x_s(\rho))) + q, \tag{14}$$

for $c \in [\underline{c}, \bar{c}]$, *where* a *and* q *are chosen satisfying*

$$a > \frac{4b_2}{(b_1)^2}, \tag{15}$$

and

$$q \in \left(\frac{b_1 b_2}{b_3} a, \frac{(b_1)^2}{4b_2} a \right), \tag{16}$$

respectively, and $\underline{c}$ *and* $\bar{c}$ *are given by*

$$\underline{c} = f(x_s(\rho)) - \frac{b_1}{2} x_s(\rho) > 0, \tag{17}$$

and

$$\bar{c} = f(x_s(\rho)) + \frac{b_1}{2} \left[x^s(\rho) - x_s(\rho) \right], \tag{18}$$

respectively.

Proof. We restrict the state space to $[0, x^s(\rho)]$ since for the problem (1) a nonconstant optimal capital path from $x \in [0, x^s(\rho)]$ monotonically converges to either 0 or $x^s(\rho)$, as shown by Akao, Kamihigashi and Nishimura (2019, Proposition 3.1) Let a and q be positive numbers and Q a negative number. The function $\tilde{H}$ in condition (c) of Theorem 1 is written as:

$$\tilde{H}(x; q, Q, x_s(\rho)) = H^*(x, q - Q(x_s(\rho) - x)) - \frac{1}{2}\rho Q x^2 \tag{19}$$

In order for $\tilde{H}$ to be well defined, $q - Q(x_s(\rho) - x) > 0$ must hold for all $x \in [0, x^s(\rho)]$. This is equivalent to

$$0 < q + Q(x^s(\rho) - x_s(\rho)). \tag{20}$$

The sub-domain of the utility function $[\underline{c}, \bar{c}]$ is chosen as the first order condition:

$$[-a(c - f(x_s(\rho))) + q] - [q - Q(x_s(\rho) - x)] = 0 \tag{21}$$

holds with $c \in [\underline{c}, \bar{c}]$ for all $x \in [0, x^s(\rho)]$ so that $H^*(x, q - Q(x_s(\rho) - x))$ is attained. That is,

$$\underline{c} \equiv f(x_s(\rho)) + \frac{Q}{a} x_s(\rho) \text{ and } \bar{c} \equiv f(x_s(\rho)) - \frac{Q}{a}\left[x^s(\rho) - x_s(\rho)\right]. \tag{22}$$

Since $\lim_{c \searrow 0} u'(c) = \infty$ and $u'(\underline{c}) < \infty$, $\underline{c} > 0$ must hold, or equivalently it must hold that

$$\frac{-Q}{a} < \frac{f(x_s(\rho))}{x_s(\rho)}. \tag{23}$$

If the triplet (a, q, Q) satisfy $a > 0, q > 0, Q < 0$ and the two inequalities (20) and (23), the conditions (a), (b), and (d) in Theorem 1 hold with $(x^*(t), c^*(t)) = (x_s(\rho), f(x_s(\rho)))$. Condition (c) in Theorem 1 is satisfied if $\partial^2 \tilde{H}/\partial x^2 \leq 0$ for all $x \in (0, x^s(\rho))$. $\partial^2 \tilde{H}/\partial x^2$ is written as

$$\begin{aligned} \partial^2 \tilde{H}/\partial x^2 &\equiv \zeta(x; \rho, a, q, Q) \\ &= -2b_3 Q x^2 + [(3b_2 + b_3 x_s(\rho))\, Q - b_3 q] x + \frac{1}{a} Q^2 - (b_2 x_s(\rho) + \rho - 2b_1) Q + b_2 q. \end{aligned} \tag{24}$$

Since ζ is a quadratic function of x and the coefficient of x^2 is $-2b_3Q > 0$, $\tilde{H}$ is concave if

$$\zeta(0; \rho, a, q, Q) < 0 \text{ and } \zeta(x^s(\rho); \rho, a, q, Q) < 0. \tag{25}$$

Since $\zeta(x; \rho, a, q, Q)$ is continuous with respect to ρ, if we find the triplet (a, q, Q) with which the two inequalities in (25) are satisfied at the limit case of $\rho = b_1$, then the inequalities are also satisfied with this triplet for ρ in a neighborhood of $\rho = b_1$. Since $x_s(b_1) = 0$, $\zeta(0; b_1, a, q, Q) < 0$ is equivalent to

$$\frac{1}{a}Q^2 + b_1Q + b_2q < 0. \tag{26}$$

Since $x^s(b_1) = 2b_2/b_3$, $\zeta(x^s(b_1); b_1, a, q, Q) < 0$ is equivalent to

$$-2b_2\left(\frac{b_2}{b_3}Q + q\right) + \zeta(0; b_1, a, q, Q) < 0. \tag{27}$$

From these, we have conditions for the triplet (a, q, Q):

$$\frac{a}{2}\left(-b_1 - \sqrt{(b_1)^2 - \frac{4b_2q}{a}}\right) < Q < \frac{a}{2}\left(-b_1 + \sqrt{(b_1)^2 - \frac{4b_2q}{a}}\right), \tag{28}$$

and

$$\frac{b_2}{b_3}Q + q > 0. \tag{29}$$

Let us choose

$$a = \frac{4b_2}{(b_1)^2}\lambda, \quad \lambda > q > \frac{4(b_2)^2}{b_1b_3}\lambda, \quad Q = -\frac{2b_2}{b_1}\lambda, \tag{30}$$

with $\lambda > 1$. Then, (28) and (29) are satisfied. We can show that (20) and (23) are satisfied with (a, q, Q) chosen as in (30). (20) holds for all $\rho \in (b_1, \rho_I)$ if

$$0 < q + (x^s(b_1) - x_s(b_1))Q = q + \frac{2b_2}{b_3}Q. \tag{31}$$

This inequality is satisfied since from (30)

$$q + \frac{2b_2}{b_3} Q > \frac{4(b_2)^2}{b_1 b_3} \lambda - \frac{2b_2}{b_3} \frac{2b_2}{b_1} \lambda = 0. \tag{32}$$

Since

$$\frac{d}{d\rho} \frac{f(x_s(\rho))}{x_s(\rho)} = \frac{f'(x_s(\rho))x_s(\rho) - f(x_s(\rho))}{x_s(\rho)^2} \frac{dx_s(\rho)}{d\rho} > 0, \tag{33}$$

$f(x_s(\rho))$ is increasing in ρ. This implies that (23) holds for all $\rho \in (b_1, \rho_I)$ if

$$\frac{-Q}{a} \leq \lim_{\rho \searrow b_1} \frac{f(x_s(\rho))}{x_s(\rho)} = b_1. \tag{34}$$

This inequality is satisfied with Q and a in (30) since

$$\frac{-Q}{a} = \frac{2b_2\lambda/b_1}{4b_2\lambda/(b_1)^2} = \frac{b_1}{2}. \tag{35}$$

We have verified that with the triplet given by (30), all conditions in Theorem 1 are satisfied. Therefore, by Theorem 1, $x_s(\rho)$ is an optimal steady state for $\rho \in (b_1, \bar{\rho})$ in a neighborhood of $\rho = b_1$. The equalities and inequalities in the statement follow from (22) and (30). ■

At the end of this note, let us show that the model in the above proposition does not satisfy the sufficient condition for $x_s(\rho)$ not to be an optimal steady state. The sufficient condition, which is proposed by Akao, Kamihigashi and Nishimura (2019, Proposition 3.1), is

$$\rho^2 - \frac{4f''(x_s(\rho))f(x_s(\rho))}{\sigma(f(x_s(\rho)))} < 0, \tag{36}$$

where σ is the elasticity of marginal utility. $\sigma(f(x_s))$ is calculated from (14) as

$$\sigma(f(x_s(\rho))) = \frac{-f(x_s(\rho))u''(f(x_s(\rho)))}{u'(f(x_s(\rho)))} = \frac{af(x_s(\rho))}{q}. \tag{37}$$

Substitute (37) and (16) into the left-hand side of (36) and we have:

$$\rho^2 - \frac{4f''(x_s(\rho))f(x_s(\rho))}{\sigma(f(x_s(\rho)))} > \rho^2 - \frac{(b_1)^2}{b_2}\sqrt{b_2^2 - 2b_3(\rho - b_1)}. \tag{38}$$

It is easily seen that the right-hand side of (38) is strictly increasing in ρ and it takes zero at $\rho = b_1$. That is, for all $\rho \in (b_1, \rho_I)$,

$$\rho^2 - \frac{4f''(x_s(\rho))f(x_s(\rho))}{\sigma(f(x_s(\rho)))} > 0, \tag{39}$$

and (36) never holds.

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