WINPEC Working Paper Series No.E1502
May 2015

A Necessary and Sufficient Condition for Non-Emptiness of the Core of a Partition Function Form Game

Takaaki Abe, Yukihiko Funaki

Waseda INstitute of Political EConomy
Waseda University
Tokyo,Japan

A Necessary and Sufficient Condition for Non-Emptiness of the Core of a Partition Function Form Game

Takaaki Abe * Yukihiko Funaki †

May 12, 2015

Abstract

The purpose of this paper is to provide a necessary and sufficient condition for nonempty core in partition function form games. We generalize the Bondareva-Shapley condition for partition function form games and present the existence conditions for the *pessimistic core* and the *optimistic core*. In addition, we study the condition for the core defined by exogenously provided partitions. The balanced collections in partition function form and some applications are also discussed.

Keywords: Cooperative games; Partition function; Core; Externalities
JEL Classification: C71

1 Introduction

The core is one of the most basic and important solutions in cooperative games. The objective of this paper is to present a necessary and sufficient condition of the nonempty core in the situation with externalities.

One of the main purposes of cooperative game theory is to illustrate how to allocate the gain obtained by a coalition to each member of the coalition. We can divide this problem into two points, namely, the amount of the gain and the distribution of the gain.

How to determine the worth of a coalition has been studied by different two models. The first model is called *coalition function form* (CFF) game in which the worth of a coalition depends only on the member of the coalition. This model has been introduced by von Neumann and Morgenstern(1953). The other one is the model allowing the worth of a coalition to depend not only on the coalition itself but also on the structure of outsiders, which is called *partition*

* Graduate School of Economics, Waseda University. E-mail: takatomo3639@asagi.waseda.jp
† Faculty of Political Science and Economics, Waseda University. E-mail: funaki@waseda,jp

function form (PFF) game. This type of formulation has been studied by Thrall (1961) and Thrall and Lucas (1963). We can distinguish these two models in terms of externalities across coalitions. CFF games are illustrating the specific situation without externalities, while PFF games are describing the situation allowing externalities. In that sense, PFF games are more general than CFF games.

The second point is the distribution of the gain. In CFF games, the core is defined as the set of distributions that are not dominated via any coalition. In order to naturally generalize this definition for PFF games, we have to make clear the concept of domination in PFF. When some players form a coalition under externalities across coalitions, members of the coalition may expect several types of reaction of outsiders. Hence, we must consider some types of domination corresponding to the type of their expectation. Indeed, unlike CFF games, some types of domination (and the corresponding core) has been proposed and studied in PFF.

In this paper, to clarify the properties of the core in PFF, we study the necessary and sufficient condition for each type of nonempty core. In the literature of the cooperative game theory, the Bondareva-Shapley condition has been known as the necessary and sufficient condition for the existence of nonempty core. This condition, however, has been applied only for CFF games. Therefore, we generalize the Bondareva-Shapley condition for PFF games and show the conditions for the several types of core in PFF.

This paper is organized as follows. In Section 2, we introduce some basic definitions and formulate partition function form games. In particular, we offer two contrastive definitions of the core, namely, the *optimistic core* and the *pessimistic core*. Furthermore, we introduce an important theorem due to Funaki and Yamato (1999) which describes the relationship between the core of PFF games and that of CFF games. Section 3 is devoted to the generalization of the Bondareva-Shapley condition for PFF games. In order to analyze the core in PFF, we compare the conditions corresponding to each type of core. Moreover, we apply our condition to the Cournot oligopoly. In Section 4, we consider the other types of core related to some exogenously provided partitions. Their existence conditions are also analyzed. In Section 5, we define the balanced collection and the minimal balanced collection in PFF. Furthermore, we apply our condition to the economic model called "the tragedy of the commons" in Section 6.

2 Preliminaries

Let $N = \{1, ..., n\}$ be a finite set of players and $n = |N|$. A partition of the player set N is defined by $\mathcal{P} = \{S_1, ..., S_h\}$ where $1 \leq h \leq n$, $S_i \neq \emptyset$ for $i = 1, ..., h$, $S_i \cap S_j = \emptyset$ for $i, j = 1, ..., h$ $(i \neq j)$, and $\bigcup_{i=1}^{h} S_i = N$. The partition consisting only of the grand coalition N is denoted by $\mathcal{P}^N$. Let Π be the set of all partitions of N and $v^{\mathcal{P}}$ be a partition function which assigns a real number $v^{\mathcal{P}}(S)$ to every coalition $S \in \mathcal{P}$. An n-person cooperative game in partition function form is defined by a triple $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$.

Throughout this paper, we assume the efficiency of the grand coalition. Formally, we assume $v^{\mathcal{P}^N}(N) > \sum_{S\in\mathcal{P}} v^{\mathcal{P}}(S)$ for any $\mathcal{P} \in \Pi \setminus \mathcal{P}^N$. [*1] [*2] Then the set of Pareto efficient payoff vectors is given by

$$E = \left\{ x \in R^N \,\middle|\, \sum_{j\in N} x_j = v^{\mathcal{P}^N}(N) \right\}.$$

As the relation between two payoff vectors in E, we consider two types of domination in partition function form studied by Funaki and Yamato(1999).

Definition 2.1. Let $x, y \in E$. We say y *pessimistically dominates* x *via* S if

$$\begin{aligned} &\sum_{j\in S} y_j \le v^{\mathcal{P}}(S) \ \textit{for all}\ \mathcal{P} \ni S;\ \textit{and} \\ &y_j > x_j \quad \textit{for all}\ j \in S. \end{aligned}$$

If there exists $S \subseteq N$ such that y pessimistically dominates x via S, then we simply say y *pessimistically dominates* x and denote by $y\ dom^{pes}\ x$.

The pessimistic domination illustrates that every member of S supposes the worst case about the structure of outsiders $N \setminus S$. If y pessimistically dominates x via S, then the coalition S can achieve the payoff vector y even in the worst case for S.

Next, we consider another type of domination that is antithetical to the pessimistic domination.

Definition 2.2. Let $x, y \in E$. We say y *optimistically dominates* x *via* S if there exists $\mathcal{P} \ni S$ such that

$$\begin{aligned} &\sum_{j\in S} y_j \le v^{\mathcal{P}}(S);\ \textit{and} \\ &y_j > x_j \quad \textit{for all}\ j \in S. \end{aligned}$$

If there exists $S \subseteq N$ such that y optimistically dominates x via S, then we simply say y *optimistically dominates* x and denote by $y\ dom^{opt}\ x$.

[*1] We can have the same discussion under the assumption $v^{\mathcal{P}^N}(N) \ge \sum_{S\in\mathcal{P}} v^{\mathcal{P}}(S)$ for any $\mathcal{P} \in \Pi \setminus \mathcal{P}^N$. However, if the equality holds, it does not imply the formation of the grand coalition (it implies the efficiency of the grand coalition). Hence the distribution of $v^{\mathcal{P}^N}(N)$ may ignore the structure of players.

[*2] Hafalir (2007) defined the convexity in PFF and proved that the convexity implies the efficiency of the grand coalition. Under this condition, we can consider the grand coalition (or the partition $\mathcal{P}^N$) will be formed.

In contrast to the pessimistic domination, the optimistic domination postulates the best reaction of outsiders. By the definitions, if y pessimistically dominates x *via* S, then y optimistically dominates x *via* the same coalition S.

By using these concepts of domination, we can consider several types of core corresponding to the each domination respectively.

Definition 2.3. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. The *pessimistic core* of $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is given by

$$C^{pes}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = \{x \in E \mid \nexists y \in E \ s.t. \ y \ dom^{pes} x\}.$$

Similarly, the *optimistic core* of $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is given by

$$C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = \{x \in E \mid \nexists y \in E \ s.t. \ y \ dom^{opt} x\}.$$

The pessimistic (optimistic) core is the set of payoff vectors that are not pessimistically (optimistically) dominated by any other payoff vector. It is easy to see that optimistic core is the subset of pessimistic core in general, because if y pessimistically dominates x then y optimistically dominates x.

Meanwhile, the core of an ordinal TU game (or CFF games without externalities) is defined as the set of payoff vectors in which any coalition S obtains the payoff not less than the value of S. In order to define a CFF game, let $N = \{1, ..., n\}$ be the player set and let $v : 2^N \to R$ be a *characteristic function* which assigns a real number to every coalition, where $v(\{\emptyset\}) = 0$. We denote a CFF game by (N, v). The *core* of (N, v) is defined by

$$C(N, v) = \left\{ x \in R^N \;\middle|\; \sum_{j\in S} x_j \geq v(S) \ \forall S \subseteq N, \ \sum_{j=1}^{n} x_j = v(N) \right\}.$$

To associate the core of a CFF game with that of a PFF game, Funaki and Yamato (1999) proved the following theorem.

Theorem 2.4 (Funaki and Yamato (1999))**.** Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. Suppose $v^{\mathcal{P}^N}(N) > \sum_{S\in\mathcal{P}} v^{\mathcal{P}}(S)$ for any $\mathcal{P} \in \Pi \setminus \mathcal{P}^N$. If we define v^{min} and v^{max} by, respectively,

$$v^{min}(S) = \min_{\mathcal{P}\ni S} v^{\mathcal{P}}(S) \ \ \forall S \subseteq N; \ and$$
$$v^{max}(S) = \max_{\mathcal{P}\ni S} v^{\mathcal{P}}(S) \ \ \forall S \subseteq N,$$

then we have

$$C^{pes}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = C(N, v^{min}); \ and$$
$$C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = C(N, v^{max}).$$

3 Necessary and sufficient conditions

In this section, we offer the necessary and sufficient condition for the core of partition function form games. First, we see the original condition for the nonempty core of CFF games, namely, the Bondareva-Shapley condition. Let (N, v) be a CFF game. Then, $C(N, v)$ is not empty if and only if

$$\begin{aligned} &for\ all\ (\delta_S)_{S\in 2^N\setminus\{\emptyset\}}\ s.t.\ \left\{ \begin{array}{l} 0 \le \delta_S \le 1 \ \ \forall S \in 2^N \setminus \{\emptyset\} \\ \sum_{S\ni i} \delta_S = 1 \ \ \forall i \in N, \end{array} \right. \\ &v(N) \ge \sum_{S\in 2^N\setminus\{\emptyset\}} \delta_S v(S). \end{aligned}$$

The original Bondareva-Shapley condition states that the core is not empty if and only if the value of the grand coalition is greater than the sum of weighted values for all coalitions. One of our main purposes is to generalize this condition for partition function form games. For this purpose, we define a useful concept.

Definition 3.1. A pair consisting of a coalition $S \subset N$ and a partition $\mathcal{P} \in \Pi$, *i.e.*, $(S, \mathcal{P})$, is said to be *embedded* if $S \in \mathcal{P}$. We define $\mathscr{L}$ as the set of all embedded pairs, that is,

$$\mathscr{L} = \{(S, \mathcal{P}) \in \left(2^N \setminus \{\emptyset\}\right) \times \Pi \mid S \in \mathcal{P}\}$$

and denote by $|\mathscr{L}|$ the number of elements of $\mathscr{L}$.

For simplicity, we denote, for example, coalition $\{1, 2\}$ by 12. In the case of $N = \{1, 2, 3\}$, the set of all embedded pairs is

$$\begin{aligned} \mathscr{L} = \{&(1, \{1, 2, 3\}),\ (1, \{1, 23\}),\ (2, \{1, 2, 3\}),\ (2, \{2, 13\}),\ (3, \{1, 2, 3\}),\ (3, \{3, 12\}), \\ &(12, \{12, 3\}),\ (13, \{13, 2\}),\ (23, \{23, 1\}),\ (123, \{123\})\}. \end{aligned}$$

Note that, for instance, $(12, \{1, 2, 3\})$ is not embedded because the coalition 12 is not the element of the partition $\{1, 2, 3\}$. In words, if a pair $(S, \mathcal{P})$ is embedded, the coalition S is formed under the partition $\mathcal{P}$. We can see the partition function as the function assigning a real number to every pair $(S, \mathcal{P}) \in \mathscr{L}$.

By using the concept of embedded pair, we can generalize the Bondareva-Shapley condition.

Proposition 3.2. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. $C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if,

$$\begin{aligned} &for\ all\ (\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}\ s.t.\ \left\{ \begin{array}{l} 0 \le \delta_S^{\mathcal{P}} \le 1 \ \ \forall (S, \mathcal{P}) \in \mathscr{L} \\ \sum_{S\ni i} \sum_{\mathcal{P}\ni S} \delta_S^{\mathcal{P}} = 1 \ \ \forall i \in N \end{array} \right. \\ &v^{\mathcal{P}^N}(N) \ge \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S). \end{aligned} \tag{3.1}$$

Proof. In view of Theorem 2.4, we have

$$
\begin{aligned}
C^{opt}(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) &= C(N,v^{max}) \\
&= \left\{ x\in R^N \,\middle|\, \sum_{j\in S} x_j \geq v^{max}(S)\ \forall S\subseteq N,\ \sum_{j\in N} x_j = v^{max}(N) \right\} \\
&= \left\{ x\in R^N \,\middle|\, \sum_{j\in S} x_j \geq \max_{\mathcal{P}\ni S} v^{\mathcal{P}}(S)\ \forall S\subseteq N,\ \sum_{j\in N} x_j = v^{\mathcal{P}^N}(N) \right\} \\
&= \left\{ x\in R^N \,\middle|\, \sum_{j\in S} x_j \geq v^{\mathcal{P}}(S)\ \forall S\subseteq N\ \forall \mathcal{P}\ni S,\ \sum_{j\in N} x_j = v^{\mathcal{P}^N}(N) \right\} \\
&= \left\{ x\in R^N \,\middle|\, \sum_{j\in S} x_j \geq v^{\mathcal{P}}(S)\ \forall (S,\mathcal{P})\in\mathscr{L},\ \sum_{j\in N} x_j = v^{\mathcal{P}^N}(N) \right\}.
\end{aligned}
$$

We consider the following linear programing problem:

$$\min \sum_{j\in N} x_j \ s.t. \ \sum_{j\in S} x_j \geq v^{\mathcal{P}}(S) \quad \forall (S,\mathcal{P})\in\mathscr{L}.$$

The optimistic core is not empty if and only if the value of this prime program is lower than $v^{\mathcal{P}^N}(N)$. Hence, the dual program is

$$\max \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S) \ s.t. \ \sum_{S\ni i}\sum_{\mathcal{P}\ni S} \delta_S^{\mathcal{P}} = 1 \quad \forall i\in N \ .$$

It follows from the duality theorem that if the prime program is feasible, then the dual program is also feasible and then both programs have the same value. Thus, the partition function form game $(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ has a nonempty optimistic core if and only if (3.1) holds. $\square$

Proposition 3.2 shows that when we sum up the weighted values for all embedded pairs, the total sum is less than the value of the grand coalition if and only if the optimistic core exists. The important difference between the original Bondareva-Shapley condition and Proposition 3.2 is the domain of the summation. In the original Bondareva-Shapley condition, we sum up the weighted values for all coalitions (except for empty set), while we sum up them for all embedded pairs in Proposition 3.2. It is interesting that the generalization of the Bondareva-Shapley condition coincide with the condition for the optimistic core other than the pessimistic core.

Next, we offer a necessary and sufficient condition for the existence of pessimistic core.

Proposition 3.3. Let $(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form. $C^{pes}(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$

is not empty if and only if,

$$
\begin{aligned}
&\exists(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}\ s.t.\ \lambda_S^{\mathcal{P}}\in\{0,1\}\ \ \forall(S,\mathcal{P})\in\mathscr{L},\ \sum_{\mathcal{P}\ni S}\lambda_S^{\mathcal{P}}=1\ \ \forall S\subseteq N\ and\\
&for\ all\ (\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}\ s.t.\ \begin{cases} 0\le\delta_S^{\mathcal{P}}\le 1\ \ \forall(S,\mathcal{P})\in\mathscr{L}\\ \sum_{S\ni i}\sum_{\mathcal{P}\ni S}\delta_S^{\mathcal{P}}\lambda_S^{\mathcal{P}}=1\ \ \forall i\in N\end{cases}\\
&v^{\mathcal{P}^N}(N)\ge\sum_{(S,\mathcal{P})\in\mathscr{L}}\delta_S^{\mathcal{P}}\lambda_S^{\mathcal{P}}v^{\mathcal{P}}(S).
\end{aligned}
\tag{3.2}
$$

Proof. As with the proof of the optimistic core, we have

$$
\begin{aligned}
C^{pes}(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) &= C(N,v^{min})\\
&=\left\{x\in R^N\ \middle|\ \sum_{j\in S}x_j\ge v^{min}(S)\ \forall S\subseteq N,\ \sum_{j\in N}x_j=v^{min}(N)\right\}\\
&=\left\{x\in R^N\ \middle|\ \sum_{j\in S}x_j\ge\min_{\mathcal{P}\ni S}v^{\mathcal{P}}(S)\ \forall S\subseteq N,\ \sum_{j\in N}x_j=v^{\mathcal{P}^N}(N)\right\}\\
&=\left\{x\in R^N\ \middle|\ \forall S\subseteq N\ \exists\mathcal{P}\ni S\ s.t.\ \sum_{j\in S}x_j\ge v^{\mathcal{P}}(S),\ \sum_{j\in N}x_j=v^{\mathcal{P}^N}(N)\right\}.
\end{aligned}
$$

Now, for all $(S,\mathcal{P})\in\mathscr{L}$ we let

$$
\lambda_S^{\mathcal{P}}=\begin{cases}1 & if\ v^{\mathcal{P}}(S)=\min_{\mathcal{P}'\ni S}v^{\mathcal{P}'}(S),\\ 0 & otherwise.\end{cases}
\tag{3.3}
$$

If there exist two or more minimums for a coalition S, we have to choose one partition, $\mathcal{P}^*$, among these partitions and let $\lambda_S^{\mathcal{P}^*}=1$. Also, for any $\mathcal{P}'\ni S$ with $\mathcal{P}'\neq\mathcal{P}^*$ we let $\lambda_S^{\mathcal{P}'}=0$. By using $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$ satisfying (3.3), we can consider the following linear programing problem:

$$
\min\sum_{j\in N}x_j\ s.t.\ \lambda_S^{\mathcal{P}}\sum_{j\in S}x_j\ge\lambda_S^{\mathcal{P}}v^{\mathcal{P}}(S)\ \ \forall(S,\mathcal{P})\in\mathscr{L}.
$$

The pessimistic core is not empty if and only if the value of this prime program is lower than $v^{\mathcal{P}^N}(N)$. Hence, the dual program is

$$
\max\sum_{(S,\mathcal{P})\in\mathscr{L}}\delta_S^{\mathcal{P}}\lambda_S^{\mathcal{P}}v^{\mathcal{P}}(S)\ s.t.\ \sum_{S\ni i}\sum_{\mathcal{P}\ni S}\delta_S^{\mathcal{P}}\lambda_S^{\mathcal{P}}=1\ \ \forall i\in N.
$$

Thus, the partition function form $(N,\Pi,\{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ has a nonempty pessimistic core if and only if there exists $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$ satisfying (3.2). □

Note that the condition of Proposition 3.2 is more strict than that of Proposition 3.3. We can see this fact by choosing a vector whose coordinate are 1, namely, $(1,...,1)\in R^{|\mathscr{L}|}$, as the vector $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$ in Proposition 3.3. If $\lambda_S^{\mathcal{P}}=1$ for any $(S,\mathcal{P})\in\mathscr{L}$, the condition (3.2) coincides with the condition (3.1). This fact indicates that the optimistic core is a subset of the pessimistic core in general.

We can interpret $\lambda_S^{\mathcal{P}}$ as the preparation of coalition S for partition $\mathcal{P}$. If the expectation of each member of S is optimistic, they foresee all partitions including S would happen. On the other hand, if the members of S have a pessimistic view, they consider that only one partition forms and that the other possible partitions would not occur.

Remark 3.4. Proposition 3.2 and Proposition 3.3 coincide with the original Bondareva-Shapley condition if for every $S \subseteq N$ and for every $\mathcal{P}, \mathcal{P}' \ni S$,

$$v^{\mathcal{P}}(S) = v^{\mathcal{P}'}(S).$$

If the foregoing equality holds, the value of each coalition does not depend on partitions. In other words, we can see the partition function form game as a coalition function form game where the value of every coalition does not depend on the partitions.

Example 3.5 (Cournot oligopoly)**.** The Cournot oligopoly model in PFF is given by Ray and Vohra (1999). We apply our analysis of optimistic and pessimistic cores to their model. We consider an industry consisting of three identical firms ($i = 1, 2, 3$) with a homogeneous output and marginal costs of production denoted by c. Let q_i denotes the quantity of the firm i. This market has the inverse demand function $p = a - bQ$, where Q is the aggregate quantity of this market. By standard calculations, the partition function form game is given by Table 1, where $M = \frac{(a-c)^2}{b}$.

Table 1 Partitions, Coalitions and Values

$\mathcal{P}$	{1,2,3}			{1,23}		{2,13}		{3,12}		$\mathcal{P}^N$
S	1	2	3	1	23	2	13	3	12	N
$v^{\mathcal{P}}(S)$	$\frac{1}{16}M$	$\frac{1}{16}M$	$\frac{1}{16}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{9}M$	$\frac{1}{4}M$

It is easy to see that this game satisfies the assumption $v^{\mathcal{P}^N}(N) > \sum_{S \in \mathcal{P}} v^{\mathcal{P}}(S)$ for any $\mathcal{P} \in \Pi \setminus \mathcal{P}^N$. This game has a nonempty pessimistic core because there exists $(\lambda_S^{\mathcal{P}}) = (\lambda_1^{\{1,2,3\}}, ..., \lambda_N^{\mathcal{P}^N}) = (1, 1, 1, 0, 1, 0, 1, 0, 1, 1)$ such that the inequality

$$\begin{aligned} & v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P}) \in \mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S) \\ \iff \quad & \frac{1}{4}M \geq \frac{1}{16}M\left(\delta_1^{\{1,2,3\}} + \delta_2^{\{1,2,3\}} + \delta_3^{\{1,2,3\}}\right) + \frac{1}{9}M\left(\delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}}\right) + \frac{1}{4}M\delta_N^{\mathcal{P}^N} \end{aligned}$$

holds for any $(\delta_S^{\mathcal{P}})$ satisfying

$$\begin{aligned}
\delta_1^{\{1,2,3\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\mathcal{P}^N} &= 1, \\
\delta_2^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\mathcal{P}^N} &= 1 \ and \\
\delta_3^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_N^{\mathcal{P}^N} &= 1.
\end{aligned}$$

On the other hand, the optimistic core is empty because, for example, we can take $(\delta_S^{\mathcal{P}})$ such that

$$\begin{aligned}
&\delta_1^{\{1,23\}} = 1,\ \delta_2^{\{2,13\}} = 1,\ \delta_3^{\{3,12\}} = 1 \ and \\
&\delta_S^{\mathcal{P}} = 0 \ for\ all\ (S,\mathcal{P})\ except\ for\ (1,\{1,23\}),(2,\{2,13\})(3,\{3,12\}).
\end{aligned}$$

This $(\delta_S^{\mathcal{P}})$ satisfies the constraints

$$\begin{aligned}
\delta_1^{\{1,2,3\}} + \delta_1^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\mathcal{P}^N} &= 1, \\
\delta_2^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_{12}^{\{3,12\}} + \delta_N^{\mathcal{P}^N} &= 1, \\
\delta_3^{\{1,2,3\}} + \delta_{23}^{\{1,23\}} + \delta_{13}^{\{2,13\}} + \delta_3^{\{3,12\}} + \delta_N^{\mathcal{P}^N} &= 1.
\end{aligned}$$

Yet, it yields the inequality

$$\begin{aligned}
& v^{\mathcal{P}^N}(N) < \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S) \\
\iff \quad & \frac{1}{4}M < \frac{1}{9}M\left(\delta_1^{\{1,23\}} + \delta_2^{\{2,13\}} + \delta_3^{\{3,12\}}\right).
\end{aligned}$$

Hence, this Cournot oligopoly model has a nonempty pessimistic core and an empty optimistic core.

4 The core based on partitions

In this section, we analyze the other types of core and their existence conditions. As we have seen, the optimistic core and the pessimistic core are poles apart in terms of the reaction of outside players. Unlike these two, in this section, we consider the core defined by exogenously provided partitions. Before the discussion of the core, we introduce a useful definition.

Notation 4.1. Let $S \subseteq N$ and $\mathcal{P} \in \Pi$. We denote by $\mathcal{P}|_S$ the *projection* of partition $\mathcal{P}$ onto coalition S, formally,

$$\mathcal{P}|_S = \{S \cap C \mid C \in \mathcal{P},\ S \cap C \neq \emptyset\}.$$

The projection of a partition onto a coalition is a partition of the coalition. The following example is useful to understand the concept of projection.

Example 4.2. Let $N = \{1,2,3,4\}$ and $\mathcal{P} = \{12,3,4\} \in \Pi$. If $S = \{123\}$, then we have $\mathcal{P}|_S = \{12,3\}$. If $S = \{234\}$ then $\mathcal{P}|_S = \{2,3,4\}$.

For simplicity, we denote $N \setminus S$ by S^c. Note that a combination of any coalition S and a projection of any partition of N onto S^c yield a partition of N, formally, for any $S \in 2^N \setminus \{\emptyset\}$ and any $\mathcal{P} \in \Pi$, we have $\{S, \mathcal{P}|_{S^c}\} \in \Pi$.

Now, we denote by $\mathcal{P}^I$ the partition consisting of singletons, *i.e.* $\mathcal{P}^I = \{\{1\}, ..., \{n\}\}$. The partition $\mathcal{P}^I$ is the finest partition in all partitions of N.

Definition 4.3. Let $x, y \in E$. We say y *$\mathcal{P}^I$-exogenously dominates x via S* if

$$\sum_{j \in S} y_j \leq v^{\{S, \mathcal{P}^I|_{S^c}\}}(S); \ \textit{and}$$
$$y_j > x_j \quad \textit{for all } j \in S.$$

If there exists $S \subseteq N$ such that y $\mathcal{P}^I$-exogenously dominates x via S, then we simply say y *$\mathcal{P}^I$-exogenously dominates* x and denote it by $y \ dom^{IEX} \ x$.

This illustrates the situation where each member of S considers that if coalition S deviates from the original partition then all coalitions except S are broken up into singletons. In other words, all players are obliged to untie their coalition if any coalition deviates. This type of reaction has been studied by Hafalir(2007), de Dlippel and Serrano (2008).

As the opposite concept, we can consider the coarsest partition, that is, $\mathcal{P}^N = \{N\}$. Analogous to the case of $\mathcal{P}^I$, the domination based on $\mathcal{P}^N$ is defined as follows.

Definition 4.4. Let $x, y \in E$. We say y *$\mathcal{P}^N$-exogenously dominates x via S* if

$$\sum_{j \in S} y_j \leq v^{\{S, \mathcal{P}^N|_{S^c}\}}(S); \ \textit{and}$$
$$y_j > x_j \quad \textit{for all } j \in S.$$

If there exists $S \subseteq N$ such that y $\mathcal{P}^N$-exogenously dominates x via S, then we simply say y *$\mathcal{P}^N$-exogenously dominates* x and denote it by $y \ dom^{NEX} \ x$.

In contrast to $\mathcal{P}^I$-exogenous domination, the players in S foresee that outsiders form their largest coalition, *i.e.*, $\{N \setminus S\}$, after the deviation of S. This type of reaction has been studied by Hafalir(2007).

We can define the core corresponding to $\mathcal{P}^I(\mathcal{P}^N)$-exogenous domination as the set of payoff vectors that are not $\mathcal{P}^I(\mathcal{P}^N)$-exogenously dominated by any other payoff vector.

Definition 4.5. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P} \in \Pi})$ be a partition function form game. The *$\mathcal{P}^I$-exogenous core* of $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P} \in \Pi})$ is given by

$$C^{IEX}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P} \in \Pi}) = \{x \in E \mid \nexists y \in E \ s.t. \ y \ dom^{IEX} x\}.$$

We can replace $\mathcal{P}^I$ by $\mathcal{P}^N$ and define *$\mathcal{P}^N$-exogenous core* in the same manner.

The following proposition is describing the relation between the core of CFF games and that of PFF games, which is parallel to Theorem 2.4.

Proposition 4.6. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. We define v^{IEX} by

$$v^{IEX}(S) = v^{\{S,\mathcal{P}^I|_{S^c}\}}(S) \ \ \forall S \subseteq N,$$
$$v^{NEX}(S) = v^{\{S,\mathcal{P}^N|_{S^c}\}}(S) \ \ \forall S \subseteq N.$$

Then we have

$$C^{IEX}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = C(N, v^{IEX}),$$
$$C^{NEX}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi}) = C(N, v^{NEX}).$$

We can obtain the same result in the case of v^{NEX}.

Proof. We omit the proof in the case of v^{NEX}. It is analogous to this proof. For simplicity, we denote $C^{IEX}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ by C^{IEX} and $C(N, v^{IEX})$ by $\mathscr{C}(v^{IEX})$.

First, we prove $C^{IEX} \subseteq \mathscr{C}(v^{IEX})$. Take any $x \in C^{IEX}$. We suppose $x \notin \mathscr{C}(v^{IEX})$. Then there exists S such that $\sum_{j\in S} x_j < v^{IEX}(S)$. Hence a payoff vector $x' \in E$ exists such that $\sum_{j\in S} x_j < \sum_{j\in S} x'_j < v^{IEX}(S) = v^{\{S,\mathcal{P}^I|_{S^c}\}}(S)$ and $x'_j > x_j$ for all $j \in S$. Thus $x' dom^{IEX} x$, which contradicts the assumption that $x \in C^{IEX}$.

Next, we show $\mathscr{C}(v^{IEX}) \subseteq C^{IEX}$. Take any $x \in \mathscr{C}(v^{IEX})$. We suppose $x \notin C^{IEX}$. Then there exist $S \subseteq N$ and $x' \in E$ such that $x' dom^{IEX} x$ via S. Hence $\sum_{j\in S} x'_j \leq v^{\{S,\mathcal{P}^I|_{S^c}\}}(S)$ and $x'_j > x_j$ for all $j \in S$. It follows that $\sum_{j\in S} x_j < \sum_{j\in S} x'_j \leq v^{\{S,\mathcal{P}^I|_{S^c}\}}(S) = v^{IEX}(S)$. This contradicts $x \in \mathscr{C}(v^{IEX})$. $\square$

According to Proposition 4.6, the coincidence of the core between CFF and PFF holds not only for the pessimistic and optimistic core also for these types of core. Hence, we obtain the following result as a corollary of Proposition 3.2.

Corollary 4.7. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form. Let $(\lambda_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$ be

$$\lambda_S^{\mathcal{P}} = \begin{cases} 1 & if\ \mathcal{P} \setminus \{S\} = \mathcal{P}^I|_{S^c}, \\ 0 & otherwise. \end{cases} \tag{4.1}$$

$C^{IEX}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if,

$$for\ all\ (\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}\ s.t. \begin{cases} 0 \leq \delta_S^{\mathcal{P}} \leq 1 \ \ \forall (S,\mathcal{P}) \in \mathscr{L} \\ \sum_{S\ni i} \sum_{\mathcal{P}\ni S} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} = 1 \ \ \forall i \in N \end{cases}$$
$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} \lambda_S^{\mathcal{P}} v^{\mathcal{P}}(S).$$

Also, we can obtain the contrastive condition for C^{NEX} by setting the following constraint:

$$\lambda_S^{\mathcal{P}} = \begin{cases} 1 & if\ \mathcal{P} \setminus \{S\} = S^c, \\ 0 & otherwise \end{cases} \tag{4.2}$$

instead of (4.1).

In view of Definition 4.3 and Definition 4.4, we can obtain the following Corollary 4.8 which describes the relationship among these types of core in PFF. Note that this relationship depends on externalities. A PFF game $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is said to have *positive externalities* if for any mutually disjoint coalitions $S, T_1, T_2 \subseteq N$, and for any partition $\mathcal{P} \in \Pi$ with $S, T_1, T_2 \in \mathcal{P}$, we have

$$v^{\{S, T_1 \cup T_2, \mathcal{P}|_{(S\cup T_1 \cup T_2)^c}\}}(S) > v^{\{S, T_1, T_2, \mathcal{P}|_{(S\cup T_1\cup T_2)^c}\}}(S).$$

Similarly, a PFF game is said to have *negative externalities* if

$$v^{\{S, T_1 \cup T_2, \mathcal{P}|_{(S\cup T_1 \cup T_2)^c}\}}(S) < v^{\{S, T_1, T_2, \mathcal{P}|_{(S\cup T_1\cup T_2)^c}\}}(S).$$

If the game has positive externalities, a coalition benefits from the merger of other coalitions. If it has negative externalities, a coalition is damaged by the merger. These notions lead to the Corollary 4.8.

Corollary 4.8. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. We denote $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ by v_{PFF}. Then, we have

$$C^{opt}(v_{PFF}) \subseteq C^{IEX}(v_{PFF}) \subseteq C^{pes}(v_{PFF}),$$
$$C^{opt}(v_{PFF}) \subseteq C^{NEX}(v_{PFF}) \subseteq C^{pes}(v_{PFF}).$$

In particular, if the game has positive externalities, we have

$$C^{NEX}(v_{PFF}) = C^{opt}(v_{PFF}) \subseteq C^{pes}(v_{PFF}) = C^{IEX}(v_{PFF}).$$

On the contrary, if the game has negative externalities, we have

$$C^{IEX}(v_{PFF}) = C^{opt}(v_{PFF}) \subseteq C^{pes}(v_{PFF}) = C^{NEX}(v_{PFF}).$$

5 Balanced and minimal balanced collections

In this section, we define the balanced collections in PFF. Furthermore, we present another form of the necessary and sufficient condition for optimistic core and pessimistic core.

Definition 5.1. A collection $\beta \subseteq \mathscr{L}$ is said to be *optimistically balanced* if positive numbers $\delta_S^{\mathcal{P}}$ exist such that

$$\forall i \in N, \sum_{S \ni i} \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} = 1. \tag{5.1}$$

The vector $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ is called a system of *optimistically balancing weights.* On the other hand, a collection $\beta \subseteq \mathscr{L}$ is said to be *pessimistically balanced* if there exist positive numbers $\delta_S^{\mathcal{P}}$ satisfying

$$\begin{aligned} &(i) \quad \forall i \in N, \sum_{S \ni i} \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} = 1, \\ &(ii) \quad \forall S \subseteq N : S \neq \emptyset,\ \exists (S,\mathcal{P}) \in \beta \ \ \textit{and} \\ &(iii) \quad \forall (S,\mathcal{P}),(T,Q)((S,\mathcal{P}) \neq (T,Q)) \in \beta,\ \mathcal{P} \neq Q \Rightarrow S \neq T. \end{aligned} \tag{5.2}$$

We call $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ a system of *pessimistically balancing weights.* A balanced collection is called *optimistically (pessimistically) minimal balanced* if it does not contain any prvariantoper optimistically (pessimistically) balanced subcollection.

We can see some relations between the two types of balanced collections.

Remark 5.2. If β is pessimistically balanced then β is optimistically balanced. Moreover, β is pessimistically balanced if and only if β is pessimistically minimal balanced, since if we can remove a pair $(S,\mathcal{P})$ from a pessimistically balanced collection β, then (ii) requires that there exists $(S,Q) \in \beta$ such that $Q \neq \mathcal{P}$, which contradicts (iii).

Example 5.3. Let $N = \{1,2,3\}$. In Table 2, $\beta_1, ..., \beta_6$ mean some examples of collection. The symbol "+" shows that the pair is the element of the collection.

Table 2 Balanced collections

$\mathcal{P}$	{1,2,3}			{1,23}		{2,13}		{3,12}		$\mathcal{P}^N$	Opt.		Pes.	
S	1	2	3	1	23	2	13	3	12	N	Balanced	Min.	Balanced	Min.
β_1	+	+	+	-	-	-	-	-	-	-	Yes	Yes		
β_2	+	+	+	+	-	-	-	-	-	-	Yes			
β_3	+	+	+	-	+	-	+	-	+	+	Yes		Yes	Yes
β_4	-	-	-	+	+	+	+	+	+	+	Yes		Yes	Yes
β_5	+	+	-	-	-	-	-	-	-	-				
β_6	+	+	+	+	+	+	+	+	+	+	Yes			

By using the definition of balanced collections, we can obtain the necessary and sufficient conditions for the nonempty core in PFF. Proposition 3.2 and (5.1) prove the following theorem.

Theorem 5.4. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. $C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if for every optimistically balanced collection β and for every vector of balancing weights $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$,

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S).$$

Similarly, Proposition 3.3 and (5.2) prove the following theorem.

Theorem 5.5. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. $C^{pes}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if there exists a pessimistically balanced collection β and a balancing weight $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ such that

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S).$$

Each of Theorem 5.4 and Theorem 5.5 is the another expression of Proposition 3.2 and Proposition 3.3 respectively. Now, we refine these conditions by using the concept of minimal balanced collection.

Let F be the set of feasible balancing weight vectors, *i.e.*,

$$F = \left\{ (\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}} \;\middle|\; \sum_{S\ni i}\sum_{\mathcal{P}\ni S} \delta_S^{\mathcal{P}} = 1 \ \forall i \in N, \quad \delta_S^{\mathcal{P}} \geq 0 \ \forall (S,\mathcal{P}) \in \mathscr{L} \right\}.$$

The following lemma due to Peleg and Sudhölter(2007) is useful to analyze the properties of F.

Lemma 5.6 (Peleg and Sudhölter(2007))**.** Let $\mathscr{P}$ be a convex polyhedral set in R^l given by

$$\mathscr{P} = \left\{ d \in R^l \;\middle|\; \sum_{j=1}^{l} d_j a_j^t \geq b^t, \ t = 1, ..., m \right\}. \tag{5.3}$$

For $d \in \mathscr{P}$ let $S(d) = \left\{ t \in \{1, ..., m\} \;\middle|\; \sum_{j=1}^{l} d_j a_j^t = b^t \right\}$. The point $d \in \mathscr{P}$ is an extreme point of $\mathscr{P}$ if and only if the system of linear equations

$$\sum_{j=1}^{l} d'_j a_j^t = b^t \quad for\ all\ t \in S(d)$$

has d as its unique solution.

Before using Lemma 5.6, we show that F is a convex polyhedral set given by (5.3).

Lemma 5.7. F is the convex polyhedral set given by (5.3).

Proof. Let $n = |N|$. We can replace d by $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$, l by $|\mathscr{L}|$ and m by $2n+|\mathscr{L}|$. We denote embedded pairs by $Z_1, ..., Z_h, ..., Z_{|\mathscr{L}|} \in \mathscr{L}$. For every $i = 1, ..., n$, we define $\phi^i \in \{0,1\}^{|\mathscr{L}|}$ by

$$\phi_h^i = \begin{cases} 1 & if \quad i \in S \ of \ Z_h \\ 0 & if \quad i \notin S \ of \ Z_h. \end{cases}$$

For every $h = 1, ..., |\mathscr{L}|$, we define $e^h \in \{0,1\}^{|\mathscr{L}|}$ by

$$e^h_j = \begin{cases} 1 & if \quad Z_j = Z_h \\ 0 & if \quad Z_j \neq Z_h. \end{cases}$$

Then, for every $t = 1, ..., 2n + |\mathscr{L}|$, we can define $a^t \in R^{|\mathscr{L}|}$ by

$$a^t = \begin{cases} \phi^t & if \quad 1 \leq t \leq n \\ -\phi^{t-n} & if \quad n+1 \leq t \leq 2n \\ e^{t-2n} & if \quad 2n+1 \leq t \leq 2n + |\mathscr{L}| \end{cases} \tag{5.4}$$

and $b^t \in R$ by

$$b^t = \begin{cases} 1 & if \quad 1 \leq t \leq n \\ -1 & if \quad n+1 \leq t \leq 2n \\ 0 & if \quad 2n+1 \leq t \leq 2n + |\mathscr{L}|. \end{cases} \tag{5.5}$$

Hence, F is the convex polyhedral set given by (5.3). $\square$

Now, we provide the following Lemma 5.8 and Lemma 5.9 to show Theorem 5.10 which is mentioned later. We denote the set of extreme points of F by $EXT(F)$.

Lemma 5.8. Let $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}} \in F$ and $\beta = \{(S,\mathcal{P}) \in \mathscr{L} \mid \delta^{\mathcal{P}}_S > 0\}$. Then $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}} \in EXT(F)$ if and only if the system of positive numbers $\delta^{\mathcal{P}}_S$ is a unique system of balancing weights for β.

Proof. By Lemma 5.6 and Lemma 5.7, $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}}$ is an extreme point of F if and only if the system of linear equations

$$(\gamma^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}} \cdot a^t = b^t \quad for\ all\ t \in S((\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}})$$

has $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\mathscr{L}}$ as its unique solution, where a^t and b^t are given by (5.4) and (5.5) respectively. Hence, we can identify $\{(S,\mathcal{P}) \in \mathscr{L} \mid \delta^{\mathcal{P}}_S > 0\}$ as the optimistically balanced collection β and the positive numbers $\delta^{\mathcal{P}}_S$ as the unique system of balancing weights for β. $\square$

Lemma 5.9. An optimistically balanced collection is optimistically minimal balanced if and only if it has a unique system of balancing weights.

Proof. The outline of the proof is similar to Peleg and Sudhölter(2007). Let β be a optimistically balanced collection and $(\delta^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\beta}$ be a balancing weight. We first show IF part. We suppose that β is not minimal, that is, there exists a optimistically balanced collection $\beta^* \subsetneq \beta$ and a balancing weight $(\delta^{*\mathcal{P}}_S)_{(S,\mathcal{P})\in\beta^*}$. Then $(\hat{\delta}^{\mathcal{P}}_S)_{(S,\mathcal{P})\in\beta}$ given by

$$\hat{\delta}^{\mathcal{P}}_S = \begin{cases} \lambda\delta^{\mathcal{P}}_S + (1-\lambda)\delta^{*\mathcal{P}}_S & if \quad (S,\mathcal{P}) \in \beta^* \\ \lambda\delta^{\mathcal{P}}_S & if \quad (S,\mathcal{P}) \in \beta \setminus \beta^* \end{cases}$$

where $0 < \lambda \leq 1$, also become the balancing weights of β. Next, we show ONLY IF part. We suppose that β has two different balancing weights, $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ and $({\delta'}_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$. Then, some $(S,\mathcal{P}) \in \beta$ exist such that ${\delta'}_S^{\mathcal{P}} > \delta_S^{\mathcal{P}}$. Moreover, let

$$k = \min\left\{ \frac{\delta_S^{\mathcal{P}}}{{\delta'}_S^{\mathcal{P}} - \delta_S^{\mathcal{P}}} \;\middle|\; {\delta'}_S^{\mathcal{P}} > \delta_S^{\mathcal{P}} \right\}.$$

Then, we can define $({\delta^*}_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ by for all $(S,\mathcal{P}) \in \beta$,

$${\delta^*}_S^{\mathcal{P}} = (1+k)\delta_S^{\mathcal{P}} - k\delta_S^{\mathcal{P}}.$$

Thus, we obtain the optimistically minimal balanced collection of β, that is, $\beta^* = \{(S,\mathcal{P}) \in \beta \mid {\delta^*}_S^{\mathcal{P}} > 0\}$. □

Finally, we obtain the necessary and sufficient condition for nonempty optimistic core.

Theorem 5.10. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. $C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if for every minimal optimistically balanced collection β and its unique balancing weight $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$,

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S).$$

Proof. In view of Theorem 5.4, $C^{opt}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if the system of inequalities

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S) \quad \forall(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}} \in F \tag{5.6}$$

holds. By the linearity of the inequalities, we have, instead of (5.6),

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S) \quad \forall(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}} \in EXT(F).$$

Thus, by Lemma 5.8 and Lemma 5.9, the proof is complete. □

Corollary 5.11. Let $(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ be a partition function form game. $C^{pes}(N, \Pi, \{v^{\mathcal{P}}\}_{\mathcal{P}\in\Pi})$ is not empty if and only if there exists a pessimistically minimal balanced collection β and a balancing weight $(\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\beta}$ such that

$$v^{\mathcal{P}^N}(N) \geq \sum_{(S,\mathcal{P})\in\beta} \delta_S^{\mathcal{P}} v^{\mathcal{P}}(S).$$

Proof. Theorem 5.5 and Remark 5.2 prove the corollary. □

6 Application to an economic model

In this section, we apply our necessary and sufficient condition to the model called "the tragedy of the commons". The PFF model of the tragedy of the commons has been studied by Funaki and Yamato (1999).

Let $N = \{1, ..., n\}$ be the set of fishermen ($n \geq 3$). We denote by x_j the amount of labor which fisherman j offers to catch fish. A production function f associates the amount of fish with the amount of labor. We assume $f(x) = x^\theta$ where θ is a parameter in $(0, 1)$. Let $q > 0$ denote the opportunity cost. A fisherman j can get the amount of fish represented by $(x_j/\sum_{i\in N} x_i)f(x_j)$. This amount is not the result of their negotiation but the technological assumptions. It is easy to see that the amount of fish fisherman j could get is depending on the amount of labor the other fishermen input. Hence, we can build a PFF game based on this model. By some computations, we obtain

$$v^{\mathcal{P}}(S) = (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}} q^{\frac{-\theta}{1-\theta}} (1 - \theta),$$

where k is the number of coalitions existing in the partition $\mathcal{P}$. Thus, in view of (3.1), the necessary and sufficient condition for the nonempty optimistic core is obtained as follows: for all $\boldsymbol{\delta} = (\delta_S^{\mathcal{P}})_{(S,\mathcal{P})\in\mathscr{L}}$ satisfying the balancedness, it holds that

$$\theta^{\frac{\theta}{1-\theta}} \geq \sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}}. \tag{6.1}$$

This condition clarifies some features of this model. To show it, we need some preparations.

Lemma 6.1. In this model, $v^{\mathcal{P}}(S)$ is monotonically decreasing for k, *i.e.*

$$\frac{\partial}{\partial k}\left[(\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}}\right] < 0.$$

Proof. This proof is a variant of Funaki and Yamato (1999). Indeed,

$$\begin{aligned}\frac{\partial}{\partial k}\left[(\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}}\right] &= \left(\frac{\theta}{\theta - 1 + k} + \frac{\theta - 2}{k}\right)(\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{-\frac{2-\theta}{1-\theta}} \frac{1}{1-\theta} \\ &= -(\theta - 2 + 2k) \cdot (\theta - 1 + k)^{\frac{2\theta-1}{1-\theta}} k^{\frac{2\theta-3}{1-\theta}} \\ &< 0\end{aligned}$$

since $k \geq 1$ and $0 < \theta < 1$. $\square$

Now, we define a function $g(\theta, \boldsymbol{\delta})$ by

$$g(\theta, \boldsymbol{\delta}) = \left[\sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}} (\theta - 1 + k)^{\frac{\theta}{1-\theta}} k^{\frac{\theta-2}{1-\theta}}\right] - \theta^{\frac{\theta}{1-\theta}}$$

and then obtain the following proposition.

Proposition 6.2. In this model, the nonempty optimistic core exists if and only if

$$n(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}-\theta^{\frac{\theta}{1-\theta}}\leq 0.$$

Proof. For any $\theta \in (0,1)$, it holds that

$$\begin{aligned}
\max_{\boldsymbol{\delta}} g(\theta,\boldsymbol{\delta}) &= \max_{\boldsymbol{\delta}}\left[\sum_{(S,\mathcal{P})\in\mathscr{L}} \delta_S^{\mathcal{P}}(\theta-1+k)^{\frac{\theta}{1-\theta}}k^{\frac{\theta-2}{1-\theta}}\right]-\theta^{\frac{\theta}{1-\theta}} \\
&= \max_{\boldsymbol{\delta}}\left[\sum_{S\in 2^N\setminus\{\emptyset\}}\sum_{\mathcal{P}\ni S} \delta_S^{\mathcal{P}}(\theta-1+k)^{\frac{\theta}{1-\theta}}k^{\frac{\theta-2}{1-\theta}}\right]-\theta^{\frac{\theta}{1-\theta}} \\
&= \max_{\boldsymbol{\delta}}\left[\left(\sum_{S\neq N,\ S\neq\{\emptyset\}} \delta_S^{\{S,N\setminus S\}}(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\right)+\left(\theta^{\frac{\theta}{1-\theta}}\delta_N^{\mathcal{P}^N}\right)\right]-\theta^{\frac{\theta}{1-\theta}} \qquad (6.2)\\
&= \max\left\{\max_{\boldsymbol{\delta}}\left(\sum_{S\neq N,\ S\neq\{\emptyset\}} \delta_S^{\{S,N\setminus S\}}(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\right),\ \max_{\boldsymbol{\delta}}\left(\theta^{\frac{\theta}{1-\theta}}\delta_N^{\mathcal{P}^N}\right)\right\}-\theta^{\frac{\theta}{1-\theta}} \\
&= \max\left\{(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\max_{\boldsymbol{\delta}}\left(\sum_{S\neq N,\ S\neq\{\emptyset\}} \delta_S^{\{S,N\setminus S\}}\right),\ \theta^{\frac{\theta}{1-\theta}}\right\}-\theta^{\frac{\theta}{1-\theta}} \\
&= \max\left\{(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}\sum_{i\in N}\delta_i^{\{i,N\setminus i\}},\ \theta^{\frac{\theta}{1-\theta}}\right\}-\theta^{\frac{\theta}{1-\theta}} \qquad (6.3)\\
&= \max\left\{n(\theta+1)^{\frac{\theta}{1-\theta}}2^{\frac{\theta-2}{1-\theta}}-\theta^{\frac{\theta}{1-\theta}},\ 0\right\}
\end{aligned}$$

where (6.2) holds by Lemma 6.1 and (6.3) is attained by $\boldsymbol{\delta}$ such that

$$\delta_S^{\mathcal{P}}=\begin{cases} 1 & if\ S=\{i\},\ \mathcal{P}=\{i,N\setminus i\}\ for\ any\ i, \\ 0 & otherwise. \end{cases}$$

This completes the proof. □

Proposition 6.2 shows that the existence of the optimistic core depends on n and θ. Figures 1, 2 and 3 are illustrating the fact. It is easy to see that for any θ there is no optimistic core if $n \geq 4$. By some computations, we can see that this model has the nonempty optimistic core only when $\theta < 0.22...$ and $n = 3$. This fact indicates that the optimistic core exists only when the size of the society is small and the technological level, *i.e.* θ, is relatively low and that the technological progress may give each player a larger incentive of deviation.

In contrast to the conditional existence of the optimistic core, Funaki and Yamato (1999) proved that the pessimistic core always exists in this model. This difference shows that not only the technology and the population but the expectation among players also affect the stability. In addition, note that this model has positive externalities. In view of Corollary 4.8, the existence condition in Proposition 6.2 can be also applied to C^{NEX}. Moreover, as the pessimistic core, C^{IEX} also always exists and C^{NEX} is the subset of C^{IEX}.

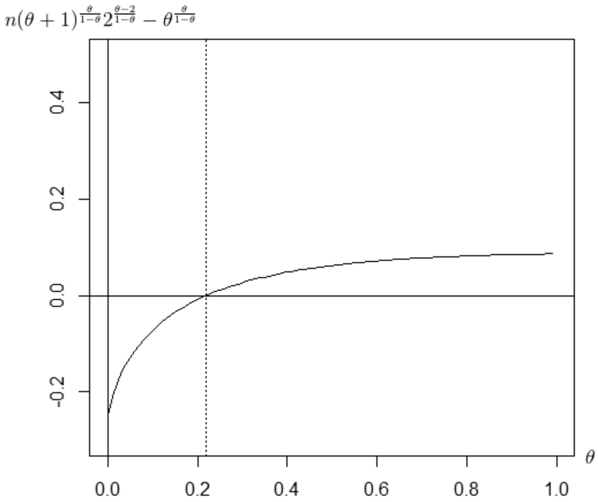

Figure 1 $n=3$

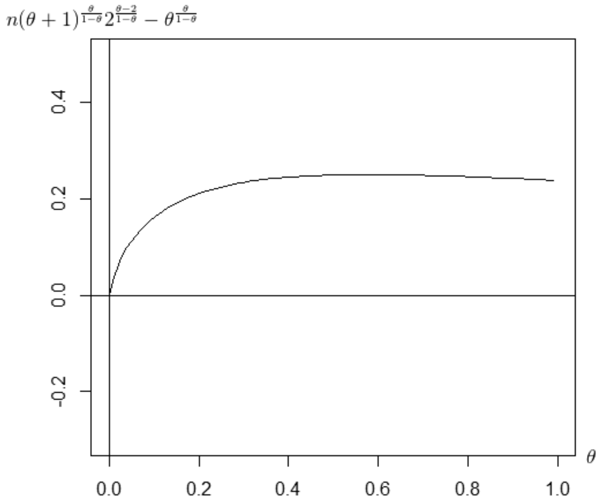

Figure 2 $n=4$

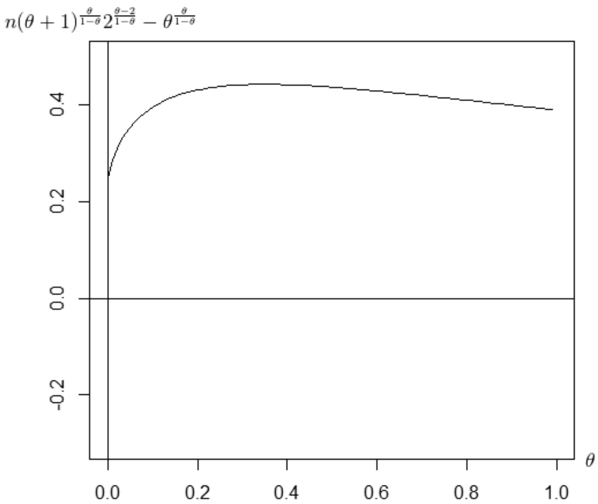

Figure 3 $n=5$

7 Conclusion

Throughout this paper, we discussed the core in PFF. In particular, we offered the generalization of the Bondareva-Shapley condition. Furthermore, we applied the generalized condition to some economic models, namely, the Cournot oligopoly and the tragedy of the commons.

In Section 4, we analyzed the two types of core based on the exogenously provided partition, *i.e.*, C^{IEX} and C^{NEX}. We can see C^{IEX} as the core based on the finest partition and C^{NEX} as the core based on the coarsest partition. We showed that their inclusion relation depends on externalities. An interesting topic is how to associate the *finer-than* relation between partitions with the inclusion relation of the core in PFF. This topic is left for future research.

For the further discussion of the stability of partition, we must relax the constraint assumed in Section 2. The constraint requires the grand coalition to be the most efficient in all embedded coalitions. Without the efficiency of the grand coalition, the payoff vectors in E do not necessarily dominate the payoff vectors out of E, where $E = \left\{ x \in R^N \middle| \sum_{j \in N} x_j = v^{\mathcal{P}^N}(N) \right\}$ as defined in Section 2. As one approach to avoid this problem, we can assume that the resulting coalition structure is $\mathcal{P}^*$ satisfying $\sum_{S \in \mathcal{P}^*} v^{\mathcal{P}^*}(S) \geq \sum_{S \in \mathcal{P}} v^{\mathcal{P}}(S)$ for any $\mathcal{P} \in \Pi$, instead of $\mathcal{P}^N$.

Reference

[1] de Clippel, G. and R. Serrano (2008) “Marginal contributions and externalities in the value”, *Econometrica* **76**, 1413-1436.

[2] Funaki, Y. and Yamato, T. (1999) “The core of an economy with a common pool resource: a partition function form approach”, *International Journal of Game Theory* **28**, 157-171.

[3] Hafalir, I. (2007) “Efficiency in coalition games with externalities”, *Games and Economic Behavior* **61**, 242-258.

[4] Peleg, B. and Sudhölter, P. (2007) *Introduction to the theory of cooperative games*, Chapter 3, Berlin: Springer.

[5] Ray, D. and Vohra, R. (1999) “A theory of endogenous coalition structures”, *Games and Economic Behavior* **26**, 286-336.

[6] Thrall, R. M. (1961) “Generalized characteristic functions for n-person games”, Proceedings of the Princeton University Conference of Oct. 1961.

[7] Thrall, R. M. and Lucas, W. F. (1963) “n-Person games in partition function form”, *Naval Research Logistics Quarterly* **10**, 281-298.

[8] von Neumann, J. and Morgenstern, O. (1953) *Theory of Games and Economic Behavior*, Princeton: Princeton University Press.