早稲田大学現代政治経済研究所

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Koji Yokote, Yukihiko Funaki, Yoshio Kamijo

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Abstract

In this paper, we investigate a new linear basis of the set of TU cooperative games. As was the case of unanimity games, the value of each game can be uniquely determined by using Efficiency, Equal Treatment Property and Null Player Property. Moreover, the linear basis induces the null space of the Shapley value. We also show that when we express a game by a linear combination of the linear basis, the Shapley value appears in the coefficients. The answer to the inverse problem of the Shapley value is given as well. Moreover, it will be clear that the coefficients can be interpreted to express a cooperative surplus regarding the Shapely value.

JEL classification: C71
Keywords: Cooperative games; Shapley value; Linear basis; Null space; Inverse problem

1 Introduction

The purpose of this paper is to investigate a new linear basis of the set of TU cooperative games.[1] The most famous linear basis is unanimity games, which were first introduced by Shapley (1953) in the proof of axiomatization. Unanimity games were useful in that the value of each game can be uniquely determined by using the following three axioms: Efficiency, Equal Treatment Property and Null Player Property. Because of this property, unanimity games have widely been used in the proof of axiomatization of the Shapley

[1] Although the linear basis was previously used in the proof of axiomatization of the weighted Shapley value by Yokote (2013), the basis itself was not discussed in detail.

value, especially in the case that the number of players is fixed throughout the proof. See, for example, Young (1985), Chun (1989) or van den Brink (2001).

Our new linear basis also satisfies this axiomatic property. Namely, each game in the linear basis can be uniquely determined by the three axioms. The difference with unanimity games is that our new linear basis contains games which span the null space of the Shapley value; the subspace in the set of all games where the Shapley value prescribes 0 vector to each game. Some previous researches have investigated the null space of the Shapley value. For instance, Kleinberg and Weiss (1985) characterized the null space by a direct-sum decomposition of the space. In their characterization of the null space, however, what games actually span the null space was unclear. Another approach was taken by Dragan, Potters and Tijs (1989). By focusing on the potential of the Shapley value, introduced by Hart and Mas-Colell (1989), they clearly defined the games which span the null space of the Shapley value. However, the value of each game in their linear basis cannot be uniquely determined only by using Efficiency, Equal Treatment Property and Null Player Property.

The difference between the previous ones and our new linear basis will be clearer from the following two properties. First, when we express a game by a linear combination of the linear basis, the coefficients can be calculated by following the same calculation of the Shapley value. In particular, the Shapley value appears in the coefficients. Second, the coefficients express a cooperative surplus of the Shapley value; let us explain this property more precisely. Consider a game and the Shapley value of player i. If some players outside of the game cooperate and the original game changes, then the Shapley value of player i will change. This change can be interpreted as the surplus of payoff for player i induced by the cooperation of players in $N\backslash S$. It will be clear that the change can be expressed by using the coefficients.

We also deal with the inverse problem, under which we characterize the set of games where the Shapley value is equal to a fixed vector. In solving the inverse problem, the linear basis players an important role. For other approaches to the problem, see Dragan (2005) and Dragan (2012).

This paper is organized as follows. Section 2 contains notations and definitions. Section 3 gives the definition of the new linear basis of the set of games. We show three ways to calculate the coefficients and prove the coincidence between the Shapley value and coefficients. In Section 4, we show that the subspace spanned by some games in the linear basis precisely coincides with the null space of the Shapley value. This result is applied to the inverse problem. In Section 5, we discuss the interpretation of the coefficients, and show that the coefficients express the cooperative surplus.

Section 6 gives concluding remarks.

2 Notations and Definitions

For any two sets A and B, $A \subset B$ means that A is a proper subset of B. $A \subseteq B$ means that $A \subset B$ or $A = B$. $|A|$ denote the cardinality of A. Let $N \subset \mathbb{N}$ be a finite set of players, and let $S \subseteq N$ be a coalition of N. We define $|N| = n$, and we restrict our attention to games with $n \geq 2$. A characteristic function $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition of N, and satisfies $v(\emptyset) = 0$. $v(S)$ can be considered to be the worth of a coalition. The pair (N, v) is called a game, and the set of all games is denoted as Γ. The set of all games with player set N is denoted as Γ^N. For notational convenience, we write v instead of (N, v) if the set of players is clear. For any $(N, v) \in \Gamma$, let $(S, v), S \subseteq N, S \neq \emptyset$ denote the restriction of (N, v) on S. A game $v \in \Gamma^N$ is called a null game if $v(S) = 0$ for all $S \subseteq N$, and denoted as v_0. A game $v \in \Gamma^N$ is called inessential if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N, S \neq \emptyset$. A game $v \in \Gamma^N$ is called 0-normalized if $v(\{i\}) = 0$ for all $i \in N$.

The Shapley value, which was first introduced by Shapley (1953), assigns a n-dimensional vector to each game $v \in \Gamma^N$ as follows:

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} \Big(v(S) - v(S \backslash \{i\}) \Big) \text{ for all } i \in N.$$

We can interpret the set Γ^N as $\mathbb{R}^{2^n - 1}$, since the worths of $2^n - 1$ coalitions must be determined. And it is known that the Shapley value ϕ is a surjective linear mapping from $\mathbb{R}^{2^n - 1}$ to $\mathbb{R}^n$. So, the mapping ϕ has a null space, which is given by

$$\{v \in \Gamma^N : \phi(v) = \mathbf{0}\}.$$

There are two real-valued functions defined on Γ which are useful to calculate the Shapley value. First, the dividend of $(N, v) \in \Gamma$, introduced by Harsanyi (1959), is defined as follows:

$$D(S, v) = \sum_{k=0}^{|S|-1} (-1)^k \sum_{T \subseteq S : |S| - |T| = k} v(T), \text{ for all } S \subseteq N, S \neq \emptyset. \tag{1}$$

Then, the following equality holds.

$$\sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T, v) = \phi_i(N, v) \text{ for all } i \in N.$$

Second, Hart and Mas-Colell (1989) introduced the potential function on Γ, which is defined recursively. For any $(N, v) \in \Gamma$,

$$P(\{i\}, v) = v(\{i\}, v), \text{ for all } i \in N.$$
$$P(S, v) = \frac{1}{|S|}\Big(v(S) + \sum_{i \in S} P(S \backslash \{i\}, v)\Big), \text{ for all } S \subseteq N, 2 \leq |S| \leq n. \quad (2)$$

Then, the following equality holds.

$$P(N, v) - P(N \backslash \{i\}, v) = \phi_i(N, v).$$

For any game $v \in \Gamma^N$, we call $i, j \in N$, $i \neq j$ as substitutes if the following condition is satisfied: $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \backslash \{i, j\}$. For any game $v \in \Gamma^N$, we call $i \in N$ as a null player if the following condition is satisfied: $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \backslash \{i\}$. For any game $v \in \Gamma^N$ and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^n$, we define the game $\alpha v + \beta \in \Gamma^N$ as follows: $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for all $S \subseteq N, S \neq \emptyset$.

We list the axioms satisfied by the Shapley value.

Efficiency $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in \Gamma^N$.

Equal Treatment Property Take any $v \in \Gamma^N$. If $i, j \in N$. $i \neq j$ are substitutes, then $\phi_i(v) = \phi_j(v)$.

Null Player Property Take any $v \in \Gamma^N$. If $i \in N$ is a null player, then $\phi_i(v) = 0$.

Linearity $\phi(\lambda v + \mu w) = \lambda \phi(v) + \mu \phi(w)$ for all $v, w \in \Gamma^N$ and $\lambda, \mu \in \mathbb{R}$.

Covariance $\psi(\alpha v + \beta) = \alpha \psi(v) + \beta$ for all $v \in \Gamma^N, \alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$.

3 New linear basis

In this section, we give a new linear basis of Γ^N. The most famous one is the set of unanimity games, which were first introduced by Shapley (1953). Unanimity game u_S for some $S \subseteq N, S \neq \emptyset$ is defined as follows:

$$u_S(T) = \begin{cases} 1 \text{ if } S \subseteq T, \\ 0 \text{ otherwise.} \end{cases}$$

When we express a game $v \in \Gamma^N$ by a linear combination of $(u_S)_{S \subseteq N, S \neq \emptyset}$, the coefficient of u_S is equal to the dividend, $D(S, v)$.

In this paper, we investigate the following set of games $(\bar{u}_S)_{S\subseteq N, S\neq\emptyset}$.

$$\bar{u}_S(T) = \begin{cases} 1 \text{ if } |T\cap S| = 1, \\ 0 \text{ otherwise.} \end{cases}$$

The game $\bar{u}_S$ states that a coalition T gets 1 if and only if T intersect with S only once. Let us interpret and compare the above two games. First, take any game $v \in \Gamma^N$ and coalition $S \subseteq N, S \neq \emptyset$, and suppose that the players in $N\backslash S$ live in a region. There is a pie which yields a payoff of 1 in the region, but the players in $N\backslash S$ cannot get the pie. The players in S are trying to go to the region and get the pie. Then, unanimity game u_S captures the situation in which the players in S can get the pie if and only if all players in S enter the region. On the other hand, the game $\bar{u}_S$ captures the situation in which a player in S can get the pie if and only if he is the first player who enters the region.

Example We give an example of $(\bar{u}_S)_{S\subseteq N, S\neq\emptyset}$ for 3-person game. Let $N = \{1, 2, 3\}$.

	$\bar{u}_{\{1\}}$	$\bar{u}_{\{2\}}$	$\bar{u}_{\{3\}}$	$\bar{u}_{\{12\}}$	$\bar{u}_{\{13\}}$	$\bar{u}_{\{23\}}$	$\bar{u}_N$
$\{1\}$	1	0	0	1	1	0	1
$\{2\}$	0	1	0	1	0	1	1
$\{3\}$	0	0	1	0	1	1	1
$\{1,2\}$	1	1	0	0	1	1	0
$\{1,3\}$	1	0	1	1	0	1	0
$\{2,3\}$	0	1	1	1	1	0	0
N	1	1	1	0	0	0	0

∎

We give a theorem and a lemma which were proved by Yokote (2013).[2]

Theorem 1 (Lemma 3 of Yokote (2013)) *The set of games $(\bar{u}_S)_{S\subseteq N, S\neq\emptyset}$ is a linear basis of Γ^N.*

This theorem was proved by using the following theorem which clarifies the relationship between the two games.

Lemma 1 (Lemma 2 of Yokote (2013)) *Let N be a set of players. Then,*

[2] Although Yokote (2013) proved the weighted version of the linear basis, the results here can be easily obtained by letting $\omega = (1, \cdots, 1)$.

we have

$$\bar{u}_S = \sum_{k=1}^{|S|} (-1)^{k-1} \sum_{T \subseteq S : |T| = k} k u_T,$$

for all $S \subseteq N$, $S \neq \emptyset$.

From Theorem 1, any game $v \in \Gamma^N$ can be expressed by a linear combination of $(\bar{u}_S)_{S \subseteq N, S \neq \emptyset}$. Let $d_N(S, v)$ be the coefficient of $\bar{u}_S, S \subseteq N, S \neq \emptyset$ in the linear combination. Namely,

$$v = \sum_{S \subseteq N : S \neq \emptyset} d_N(S, v) \bar{u}_S.$$

We show three ways to calculate the coefficients. In Theorems 2 and 3, we use the dividend, the potential, respectively. In Theorem 4, we show direct calculation from $v \in \Gamma^N$. The point is that, for any $v \in \Gamma^N$, the Shpaley value of player i coincides with $d_N(\{i\}, v)$.

We first prove the theorem which uses the dividend.

Theorem 2 *Let* (N, v) *be a game. Then, we have*

$$d_N(S, v) = (-1)^{|S|-1} \sum_{T : S \subseteq T} \frac{1}{|T|} D(T, v),$$

for all $S \subseteq N, S \neq \emptyset$. *In particular,*

$$d_N(\{i\}, v) = \sum_{T \subseteq N : i \in T} \frac{1}{|T|} D(T, v) = \phi_i(N, v),$$

for all $i \in N$.

Proof. From the assumption, we have

$$\begin{aligned}
v &= \sum_{S \subseteq N : S \neq \emptyset} d_N(S, v) \bar{u}_S \\
&= \sum_{S \subseteq N : S \neq \emptyset} d_N(S, v) \sum_{k=1}^{|S|} (-1)^{k-1} \sum_{T \subseteq S : |T| = k} k u_T \\
&= \sum_{S \subseteq N : S \neq \emptyset} \sum_{T : S \subseteq T} (-1)^{|S|-1} d_N(T, v) |S| u_S, \\
v &= \sum_{S \subseteq N : S \neq \emptyset} \frac{D(S, v)}{|S|} \cdot |S| u_S,
\end{aligned}$$

where the second equality holds from Lemma 1. From the uniqueness of $d_N(S,v), S \subseteq N, S \neq \emptyset$, we have

$$\sum_{T:S\subseteq T} (-1)^{|S|-1} d_N(T,v) = \frac{D(S,v)}{|S|}, \tag{3}$$

for all $S \subseteq N, S \neq \emptyset$.

Let us reexamine the statement of the theorem. First, let $S = N$. From equation (3), we have $(-1)^{n-1} d_N(N,v) = \frac{D(N,v)}{n}$. Or, equivalently,

$$d_N(N,v) = (-1)^{n-1} \frac{D(N,v)}{n},$$

and the statement holds. Suppose that the statement holds for all S such that $|S| \geq s+1$ and we show the result for the case of $|S| = s, 1 \leq s \leq n-1$. From equation (3),

$$\begin{aligned}\frac{D(S,v)}{s} &= (-1)^{s-1} \sum_{T:S\subseteq T} d_N(T,v) \\ &= (-1)^{s-1} \sum_{k=s}^{n} \sum_{T:S\subseteq T,|T|=k} d_N(T,v) \\ &= (-1)^{s-1} d_N(S,v) + (-1)^{s-1} \sum_{k=s+1}^{n} \sum_{T:S\subseteq T,|T|=k} d_N(T,v) \\ &= (-1)^{s-1} d_N(S,v) + (-1)^{s-1} \sum_{k=s+1}^{n} \sum_{T:S\subseteq T,|T|=k} (-1)^{k-1} \sum_{R:T\subseteq R} \frac{D(R,v)}{|R|},\end{aligned} \tag{4}$$

where the fourth equality holds from the induction hypothesis. Consider the second term of the right-hand side.[3]

$$
\begin{aligned}
&(-1)^{s-1}\sum_{k=s+1}^{n}\sum_{T:S\subseteq T,|T|=k}(-1)^{k-1}\sum_{R:T\subseteq R}\frac{D(R,v)}{|R|}\\
=&(-1)^{1-s}\sum_{R:S\subset R}\sum_{k=1}^{|R|-s}\binom{|R|-s}{k}(-1)^{s+k-1}\frac{D(R,v)}{|R|}\\
=&\sum_{R:S\subset R}\sum_{k=1}^{|R|-s}\binom{|R|-s}{k}(-1)^{k}\frac{D(R,v)}{|R|}\\
=&-\sum_{R:S\subset R}\frac{D(R,v)}{|R|},
\end{aligned}
$$

where the third equality holds from the binomial theorem. By substituting this equation into equation (4), and rearranging, we have

$$(-1)^{s-1}d_N(S,v)=\sum_{R:S\subseteq R}\frac{D(R,v)}{|R|},$$

which completes the proof. □

Next, we prove the theorem which uses the potential.

Theorem 3 *Let (N,v) be a game. Then, we have*

$$d_N(S,v)=\sum_{T:N\backslash S\subseteq T}(-1)^{|T\cap S|-1}P(T,v),$$

for all $S\subseteq N, S\neq\emptyset$. In particular,

$$d_N(\{i\},v)=P(N,v)-P(N\backslash\{i\})=\phi_i(v),$$

for all $i\in N$.

In order to prove this theorem, we need three lemmas. First, let us introduce the set of games $\tilde{u}_S, S\subseteq N, S\neq\emptyset$.

$$\tilde{u}_S(T)=\begin{cases}|S| & \text{if } T=S,\\ -1 & \text{if } T=S\cup\{j\} \text{ for some } j\in N\backslash S,\\ 0 & \text{otherwise.}\end{cases}$$

[3]We explain the first equality below. In order for $\frac{D(R,v)}{|R|}$ to be added, a coalition R' such that $R'\supset S$, $R'\subseteq R$ must be chosen. Such a coalition R' is determined by choosing k, $1\leq k\leq|R\backslash S|$, players from $R\backslash S$.

This set of games was first introduced by Dragan, Potters and Tijs (1989). Here, we use this set only for the purpose of calculating the coefficients.

Lemma 2 *Let (N, v) be a game and suppose that $v = \sum_{S \subseteq N : S \neq \emptyset} \tilde{\beta}_S \tilde{u}_S$ for some $(\beta_S)_{S \subseteq N, S \neq \emptyset}$. Then, we have*

$$\tilde{\beta}_S = P(S, v),$$

for all $S \subseteq N, S \neq \emptyset$.

Proof. From the definition, $\tilde{\beta}_{\{i\}} = v(\{i\}) = P(\{i\}, v)$. Suppose that the statement holds for $|S| = s - 1$, and we prove the case of $|S| = s, s \geq 2$. First, from the definition, we have

$$v(S) = s\tilde{\beta}_S - \sum_{i \in S} \tilde{\beta}_{S \setminus \{i\}}.$$

From the induction hypothesis,

$$v(S) = s\tilde{\beta}_S - \sum_{i \in S} P(S \setminus \{i\}, v).$$

Then, we have

$$\tilde{\beta}_S = \frac{1}{s}\left(v(S) + \textstyle\sum_{i \in S} P(S \setminus \{i\}, v)\right) = P(S, v),$$

as desired. □

Lemma 3 *Let (N, v) be a game. Then, we have*

$$\sum_{T : S \subseteq T} \tilde{u}_T = |S| u_S,$$

for all $S \subseteq N, S \neq \emptyset$.

Proof. Take any $S \subseteq N, S \neq \emptyset$ and fix. Let us calculate $\sum_{T : S \subseteq T} \tilde{u}_T(R), R \subseteq N, R \neq \emptyset$. If $S \nsubseteq R$, the value is 0.[4] If $S \subseteq R$, R gains $|R|$ from $\tilde{u}_R(R)$, but R loses 1 by $|R| - |S|$ times. Summing up, we have

$$\sum_{T : S \subseteq T} \tilde{u}_T(R) = \begin{cases} |R| - (|R| - |S|) = |S| & \text{if } R \supseteq S, \\ 0 & \text{otherwise,} \end{cases}$$

which is equal to $|S| u_S$. □

[4]From the definition of $\tilde{u}_T$, $\tilde{u}_T(R) \neq 0$ only if $R \supseteq T$. Note that if $R \not\supseteq S$, then $R \not\supseteq T$ for all T such that $T \supseteq S$. It follows that $\sum_{T : S \subseteq T} \tilde{u}_T(R)$ must be 0.

Lemma 4 *Let (N, v) be a game. Then, we have*

$$\frac{D(S,v)}{|S|} = \sum_{k=0}^{|S|-1} (-1)^k \sum_{T \subseteq S: |T| = |S|-k} P(T,v),$$

for all $S \subseteq N, S \neq \emptyset$.

Proof. From Lemmas 2 and 3, we have

$$\sum_{S \subseteq N: S \neq \emptyset} \frac{D(S,v)}{|S|} \sum_{T: S \subseteq T} \tilde{u}_T = \sum_{S \subseteq N: S \neq \emptyset} P(S,v) \tilde{u}_S,$$

$$\sum_{S \subseteq N: S \neq \emptyset} \sum_{T \subseteq S: T \neq \emptyset} \frac{D(T,v)}{|T|} \tilde{u}_S = \sum_{S \subseteq N: S \neq \emptyset} P(S,v) \tilde{u}_S.$$

From the uniqueness of the coefficients, we have

$$\sum_{T \subseteq S: T \neq \emptyset} \frac{D(T,v)}{|T|} = P(S,v), \tag{5}$$

for all $S \subseteq N, S \neq \emptyset$. Equivalently,

$$\frac{D(S,v)}{|S|} + \sum_{T \subset S: T \neq \emptyset} \frac{D(T,v)}{|T|} = P(S,v). \tag{6}$$

Let us reexamine the statement of Lemma 4. If $|S| = 1$, the proof is obvious from equation (5). Suppose that the proof holds for $|S| \leq s-1$, and we prove the case of $|S| = s, s \geq 2$. Then, from the induction hypothesis, equation (6) reduces to

$$P(S,v) = \frac{D(S,v)}{s} + \sum_{T \subset S: T \neq \emptyset} \left(\sum_{k=0}^{|T|-1} (-1)^k \sum_{R \subseteq T: |R| = |T|-k} P(R,v) \right).$$

By rearranging, we have

$$\frac{D(S,v)}{s} = P(S,v) - \sum_{T \subset S: T \neq \emptyset} \left(\sum_{k=0}^{|T|-1} (-1)^k \sum_{R \subseteq T: |R| = |T|-k} P(R,v) \right). \tag{7}$$

Consider the second term of the right-hand side. Take any $T' \subset S$. Then the coefficient of $P(T', v)$ is given by[5]

$$- \sum_{k=0}^{s-|T'|-1} (-1)^k \binom{s - |T'|}{k}.$$

[5] In order for $P(T', v)$ to be added, a coalition T'' such that $T'' \supseteq T', T'' \subset S$ is chosen in the first summation $\sum_{T \subset S: T \neq \emptyset}$. Such a coalition T'' is determined by choosing k, $0 \leq k \leq s - |T'| - 1$, players from $S \backslash T'$, since $T'' = T' \cup R$ for some $R \subset S \backslash T'$.

From the binomial theorem, we have

$$0 = (1-1)^{s-|T'|} = \sum_{k=0}^{s-|T'|} (-1)^k \binom{s-|T'|}{k},$$

$$-\sum_{k=0}^{s-|T'|-1} (-1)^k \binom{s-|T'|}{k} = (-1)^{s-|T'|}.$$

As a result, equation (7) reduces to

$$\begin{aligned}\frac{D(S,v)}{s} &= P(S,v) + \sum_{T\subset S:T\neq\emptyset} (-1)^{s-|T|} P(T,v),\\ &= P(S,v) + \sum_{k=1}^{s-1}(-1)^k \sum_{T\subset S:|T|=s-k} P(T,v),\\ &= \sum_{k=0}^{s-1}(-1)^k \sum_{T\subseteq S:|T|=s-k} P(T,v),\end{aligned}$$

as desired. □

Proof of the theorem. From Theorem 2 and Lemma 4, we have

$$\begin{aligned}d_N(S,v) &= (-1)^{|S|-1} \sum_{R:S\subseteq R} \frac{D(R,v)}{|R|}\\ &= (-1)^{|S|-1} \sum_{R:S\subseteq R} \sum_{k=0}^{|R|-1} (-1)^k \sum_{T\subseteq R:|T|=|R|-k} P(T,v), \qquad (8)\end{aligned}$$

for all $S \subseteq N, S \neq \emptyset$. Let us fix $S \subseteq N, S \neq \emptyset$ and consider the right-hand side of equation (8). Take any $T \subseteq N, T \neq \emptyset$, and we calculate the coefficient of $P(T,v)$. It is given by[6]

$$\begin{aligned}&(-1)^{|S|-1} \sum_{k=0}^{|N\backslash(S\cup T)|} \binom{|N\backslash(S\cup T)|}{k} (-1)^{|S\backslash T|+k}\\ =&(-1)^{|T\cap S|-1} \sum_{k=0}^{|N\backslash(S\cup T)|} \binom{|N\backslash(S\cup T)|}{k} (-1)^k.\end{aligned}$$

[6]Note that in order for $P(T,v)$ to be added, a coalition R such that $S \subseteq R$, $T \subseteq R$ must be chosen. Namely, by choosing k players from $N\backslash(S\cup T)$, R is determined. For such coalition R, we have $|R\backslash T| = |S\backslash T| + k$.

In the transformation above, note that $(-1)^{|S|-1} \cdot (-1)^{|S\backslash T|+k}$ is equal to $(-1)^{|S|-1} \cdot (-1)^{-|S\backslash T|} \cdot (-1)^k$, and we get $(-1)^{|T\cap S|-1} \cdot (-1)^k$. From the binomial theorem, $P(T, v) = 0$ for all T such that $S \cup T \neq N$. If $S \cup T = N$, $|N\backslash(S \cup T)| = 0$. As a result, equation (8) reduces to

$$d_N(S, v) = \sum_{T:N\backslash S\subseteq T} (-1)^{|T\cap S|-1} P(T, v),$$

for all $S \subseteq N, S \neq \emptyset$. □

Finally, we show a direct calculation.

Theorem 4 *Let (N, v) be a game. Then, we have*

$$d_N(S, v) = (-1)^{|S|-1} \sum_{R\subseteq N} (-1)^{|S\backslash R|} \frac{(n - |S \cup R|)!(|S \cup R| - 1)!}{n!} v(R),$$

for all $S \subseteq N, S \neq \emptyset$. In particular,

$$d_N(\{i\}, v) = \sum_{S\subseteq N:i\in S} \frac{(n - |S|)!(|S| - 1)!}{n!} \Big(v(S) - v(S\backslash\{i\})\Big) = \phi_i(N, v),$$

for all $i \in N$.

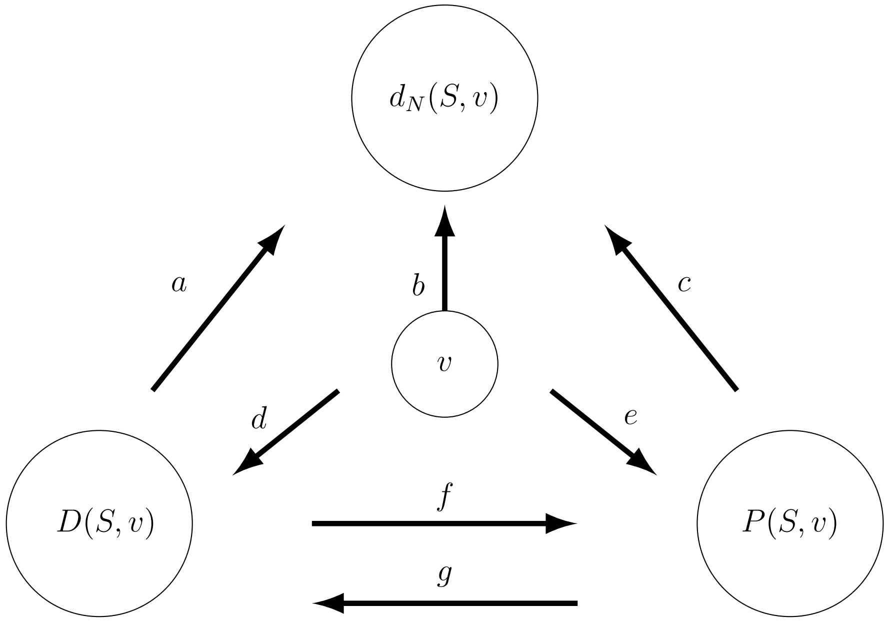
In the calculation, we follow the way proposed by Owen (1972).

Proof. Take any $v \in \Gamma^N$ and $S \subseteq N, S \neq \emptyset$. From Theorem 2,

$$
\begin{aligned}
&(-1)^{|S|-1} d_N(S,v) \\
&= \sum_{T:S\subseteq T} \frac{1}{|T|} \sum_{k=0}^{|T|-1} (-1)^k \sum_{R\subseteq T:|T|-|R|=k} v(R) \\
&= \sum_{R\subseteq N} \sum_{k=0}^{|N\setminus(R\cup S)|} \binom{|N\setminus(R\cup S)|}{k} (-1)^{k+|S\setminus R|} \frac{1}{k+|R\cup S|} v(R) \\
&= \sum_{R\subseteq N} \sum_{k=0}^{|N\setminus(R\cup S)|} (-1)^{k+|S\setminus R|} \binom{|N\setminus(R\cup S)|}{k} v(R) \int_0^1 x^{k+|R\cup S|-1} dx \\
&= \sum_{R\subseteq N} \int_0^1 \sum_{k=0}^{|N\setminus(R\cup S)|} (-1)^{k+|S\setminus R|} \binom{|N\setminus(R\cup S)|}{k} v(R) x^{k+|R\cup S|-1} dx \\
&= \sum_{R\subseteq N} \int_0^1 (-1)^{|S\setminus R|} x^{|R\cup S|-1} v(R) \sum_{k=0}^{|N\setminus(R\cup S)|} \binom{|N\setminus(R\cup S)|}{k} (-x)^k dx \\
&= \sum_{R\subseteq N} (-1)^{|S\setminus R|} v(R) \int_0^1 x^{|R\cup S|-1} (1-x)^{|N\setminus(R\cup S)|} dx \\
&= \sum_{R\subseteq N} (-1)^{|S\setminus R|} v(R) \Big(\Big[\tfrac{x^{|R\cup S|}}{|R\cup S|} (1-x)^{|N\setminus(R\cup S)|} \Big]_0^1 \\
&+ \frac{|N\setminus(R\cup S)|}{|R\cup S|} \int_0^1 x^{|R\cup S|} (1-x)^{|N\setminus(R\cup S)|-1} dx \Big) \\
&= \sum_{R\subseteq N} (-1)^{|S\setminus R|} v(R) \\
&\Big(\frac{|N|-|R\cup S|}{|R\cup S|} \cdot \frac{|N|-|R\cup S|-1}{|R\cup S|+1} \cdot \cdots \cdot \frac{1}{n-1} \int_0^1 x^{n-1} dx \Big) \\
&= \sum_{R\subseteq N} (-1)^{|S\setminus R|} \frac{(n-|R\cup S|)!(|R\cup S|-1)!}{n!} v(R).
\end{aligned}
$$

□

Before we finish this section, we summarize the results. The following figure shows how the values are correlated.



a : Theorem 2	b : Theorem 4	c : Theorem 3
d : Equation (1)	e : Equation (2)	f : Equation (5)
g : Lemma 4		

In the usual setting of TU cooperative games, our main interest is to calculate the Shapley value of some game $v \in \Gamma^N$. The Shapley value, $\phi_i(v)$, is equal to $d_N(\{i\}, v)$, which appears in the circle placed on the top. So, this figure generalizes the process of calculation, from the Shapley value to the coefficients.

Moreover, we can calculate the worths of coalitions of game $v \in \Gamma^N$ given the coefficients $d_N(S, v)$. Namely, the inverse arrow of b can be obtained.[7]

$$v(S) = \sum_{T \subseteq N : |T \cap S| = 1} d_N(T, v), \tag{9}$$

for all $S \subseteq N, S \neq \emptyset$. It follows that if the value of one circle is obtained, then all other values can also be obtained.

4 Inverse Problem

In this section, we focus on the inverse problem of the Shapley value. For a n-dimensional vector x, we characterize the set of games where the Shapley

[7] Note that only the games $\bar{u}_T$, $T \subseteq N$, $T \neq \emptyset$, such that $|T \cap S| = 1$ gives 1 to coalition S in the linear combination.

value is exactly x. By solving this problem, we can clarify the set of games where the Shapley value prescribes the same vector. The answer also tells us the invariance of the Shapley value against the change of the structure of games.

Before we tackle with the inverse problem, we first prove the important property of the linear basis. As we saw, the coefficients of games $\{\bar{u}_{\{i\}} : i \in N\}$ are exactly the Shapley value. On the other hand, other games in the linear basis span the null space of the Shapley value.

Theorem 5 *The subspace spanned by $\bar{u}_S, S \subseteq N, |S| \geq 2$ is the null space of ϕ.*

We fist prove a lemma.

Lemma 5 *For any set of players N, we have $\phi(\bar{u}_S) = \mathbf{0}$ for all $S \subseteq N, |S| \geq 2$.*

Proof. Take any $\bar{u}_S, |S| \geq 2$. If $|S| = n$, all players are substitutes. From Equal Treatment Property, $\phi_i(v) = \phi_j(v)$ for all $i, j \in N$. Combined with Efficiency, we have $\phi(v) = \mathbf{0}$.

Suppose that $2 \leq |S| \leq n-1$ and take any $k \in N \backslash S$. Since $T \cap S = (T \cup \{k\}) \cap S$ for all $T \subseteq N \backslash \{k\}$, from the definition of $\bar{u}_S$, k is a null player. From Null Player Property, $\phi_k(v) = 0$ for all $k \in N \backslash S$. Consider $i \in S$. Since all players in S are substitutes, from Equal Treatment Property, we have $\phi_i(v) = \phi_j(v)$ for all $i, j \in S$. Combined with Efficiency and the fact that $\phi_k(v) = 0$ for all $k \in N \backslash S$, we have $\phi(v) = \mathbf{0}$, which completes the proof. □

Proof of the theorem. Since ϕ is a surjective linear mapping, we already know that the dimension of the null space is $2^n - 1 - n$. As we saw in Theorem 1, $(\bar{u}_S)_{S \subseteq N, |S| \geq 2}$ are linearly independent. Lemma 5 completes the proof.

□

From Lemma 5 and Linearity, the change in the coefficients of $\bar{u}_S$, $|S| \geq 2$, namely the change in $d_N(S, v), |S| \geq 2$, has nothing to do with the Shapley value. In other words, we can change the value of $d_N(S, v), |S| \geq 2$ without changing the Shapley value. This viewpoint is the key in the proof of the next theorem.

Theorem 6 *Let (N, v) be a game. Then, $x \in \mathbb{R}^n$ is the Shapley value of*

(N, v) *if and only if there exists* $(y_S)_{S\subseteq N,|S|\geq 2} \in \mathbb{R}^{2^n-1-n}$ *such that*

$$v(S) = \sum_{i\in S} x_i + \sum_{T\subseteq N:|T|\geq 2,|T\cap S|=1} y_T,$$

for all $S \subseteq N, S \neq \emptyset$.

Proof. **Only if part.** Let $y_T = d_N(T, v)$ for all $T \subseteq N, |T| \geq 2$ and let $x_i = \phi_i(N, v)$ for all $i \in N$. Then, the equation in the statement holds for all $S \subseteq N, S \neq \emptyset$ from equation (9), which completes the proof.

If part. We first prove a lemma.

Lemma 6 *Let* $\psi : \Gamma^N \to \mathbb{R}^n$ *be a solution function which satisfies Covariance and Null Player Property. Let* $v \in \Gamma^N$ *be an inessential game. If* $\psi(v) = \mathbf{0}$, *then* v *is a null game.*

Proof. Consider the game $v - \beta, \beta \in \mathbb{R}^n$, where $\beta_i = v(\{i\}), i = 1, \dots, n$. Then, from the definition of inessential game, $v - \beta$ is a null game. From Null Player Property, we have

$$\psi_i(v - \beta) = 0,$$

for all $i \in N$. From Covariance,

$$0 = \psi_i(v - \beta) = \psi_i(v) - \beta_i = -\beta_i,$$

it follows that $v(\{i\}) = 0$ for all $i \in N$. Namely, the game v is a null game. $\square$

Suppose that $\phi(N, v) = \tilde{x} \neq x$. From Theorem 1, we can express v by a linear combination of $(\bar{u}_S)_{S\subseteq N,S\neq\emptyset}$. Moreover from Theorem 2 (or 3, 4), the coefficients of the first n games coincide with the Shapley value.

$$v = \sum_{i\in N} \tilde{x}_i d_N(\{i\}, v) + \sum_{T\subseteq N:|T|\geq 2} d_N(T, v)\bar{u}_T.$$

We define $v' := v + \sum_{T\subseteq N:|T|\geq 2}\{y_T - d_N(T, v)\}\bar{u}_T$. Then, from Lemma 5 and Linearity, $\phi(v') = \tilde{x}$. The game v' is given by

$$v' = \sum_{i\in N} \tilde{x}_i d_N(\{i\}, v) + \sum_{T\subseteq N:|T|\geq 2} y_T\bar{u}_T.$$

Consider the worth of coalition in v'.

$$v'(S) = \sum_{i\in S} \tilde{x}_i + \sum_{T\subseteq N:|T|\geq 2,|T\cap S|=1} y_T \text{ for all } S \subseteq N, S \neq \emptyset. \tag{10}$$

And we already know from the assumption that

$$v(S) = \sum_{i \in S} x_i + \sum_{T \subseteq N : |T| \geq 2, |T \cap S| = 1} y_T \text{ for all } S \subseteq N, S \neq \emptyset. \tag{11}$$

Consider the game $v - v' \in \Gamma^N$. From equation (10) and (11), we have

$$(v - v')(S) = \sum_{i \in S} (x_i - \tilde{x}_i) \text{ for all } S \subseteq N, S \neq \emptyset.$$

Moreover, from Linearity, we have

$$\phi(v - v') = \phi(v) - \phi(v') = \tilde{x} - \tilde{x} = \mathbf{0}.$$

Since $v - v'$ is an inessential game, Lemma 6 implies that $v - v'$ is a null game. In particular,

$$(v - v')(\{i\}) = x_i - \tilde{x}_i = 0 \text{ for all } i \in N.$$

It follows that $x = \tilde{x}$, which shows the desired contradiction.

□

Example We give an example of the inverse problem for 3-person game. Let $N = \{1, 2, 3\}$. The set of all games v such that $\phi(v) = x$ can be expressed by using $(y_S)_{S \subseteq N, S \neq \emptyset}$ as follows.

$$\begin{aligned}
v(\{1\}) &= x_1 + y_{12} + y_{13} + y_N, \\
v(\{2\}) &= x_2 + y_{12} + y_{23} + y_N, \\
v(\{3\}) &= x_3 + y_{13} + y_{23} + y_N, \\
v(\{1,2\}) &= x_1 + x_2 + y_{13} + y_{23}, \\
v(\{1,3\}) &= x_1 + x_3 + y_{12} + y_{23}, \\
v(\{2,3\}) &= x_2 + x_3 + y_{12} + y_{13}, \\
v(N) &= x_1 + x_2 + x_3.
\end{aligned}$$

The set of all 0-normalized games v such that $\phi(v) = x$ can be expressed by using $y \in \mathbb{R}$ as follows.

$$\begin{aligned}
v(\{1,2\}) &= x_1 + x_2 - x_3 - y, \\
v(\{1,3\}) &= x_1 + x_3 - x_2 - y, \\
v(\{2,3\}) &= x_2 + x_3 - x_1 - y, \\
v(\{N\}) &= x_1 + x_2 + x_3.
\end{aligned}$$

Let us explain the equations above. First, each coalition S gets the sum of amounts which players in S receive in x. After that, the value $-y$ is added. The change in y has nothing to do with the Shapley value x, since all players are symmetric in the change and the worth of the grand coalition never changes. ∎

5 The Shapley value of subgames and Cooperative Surplus

In this section, we state the final property of the new linear basis. That is, the coefficients express the cooperative surplus of the Shapley value. In order to explain this property in detail, we need to prove the main theorem of this section, and the proof needs two lemmas.

Let us remark the notation of this section. We write $\phi(N, v)$ instead of $\phi(v)$ and write $D_N(S, v)$ instead of $D(S, v)$, in order to explicitly refer to the domain of a game v.

Lemma 7 *Let (N, v) be a game and let $j \in N$. Then, we have*

$$d_{N\backslash\{j\}}(S, v) = d_N(S, v) + d_N(S \cup \{j\}, v),$$

for all $S \subseteq N\backslash\{j\}, S \neq \emptyset$.

Proof. Take any $S \subseteq N\backslash\{j\}, S \neq \emptyset$. From Theorem 2, we have

$$\begin{aligned} d_N(S, v) &= (-1)^{|S|-1} \sum_{T \subseteq N : S \subseteq T} \frac{1}{|T|} D_N(T, v) \\ &= (-1)^{|S|-1} \sum_{T \subseteq N\backslash\{j\} : S \subseteq T} \frac{1}{|T|} D_N(T, v) \\ &\quad + (-1)^{|S|-1} \sum_{T \subseteq N : S \cup \{j\} \subseteq T} \frac{1}{|T|} D_N(T, v), \\ d_N(S \cup \{j\}, v) &= (-1)^{|S|} \sum_{T \subseteq N : S \cup \{j\} \subseteq T} \frac{1}{|T|} D_N(T, v). \end{aligned}$$

Since

$$(-1)^{|S|-1} \sum_{T \subseteq N : S \cup \{j\} \subseteq T} \frac{1}{|T|} D_N(T, v) + (-1)^{|S|} \sum_{T \subseteq N : S \cup \{j\} \subseteq T} \frac{1}{|T|} D_N(T, v) = 0,$$

we must have

$$d_N(S,v) + d_N(S \cup \{j\}, v) = (-1)^{|S|-1} \sum_{T \subseteq N\backslash\{j\}: S \subseteq T} \frac{1}{|T|} D_N(T,v).$$

From the definition of the dividend, we have $D_N(T,v) = D_{N\backslash\{j\}}(T,v)$ for all $T \subseteq N\backslash\{j\}$. By substituting this equation, we have

$$d_N(S,v) + d_N(S \cup \{j\}, v) = (-1)^{|S|-1} \sum_{T \subseteq N\backslash\{j\}: S \subseteq T} \frac{1}{|T|} D_{N\backslash\{j\}}(T,v),$$

which is equal to $d_{N\backslash\{j\}}(S,v)$ from Theorem 2. □

The following lemma is an extension of Lemma 7.

Lemma 8 *Let (N,v) be a game and let $T \subset N, T \neq \emptyset$. Then, we have*

$$d_{N\backslash T}(S,v) = \sum_{R \in 2^T} d_N(S \cup R, v),$$

for all $S \subseteq N\backslash T, S \neq \emptyset$.

Proof. From Lemma 7, the statement holds if $|T| = 1$. Suppose that the statement holds for all T such that $|T| = k-1$, and we prove the case of $|T| = k$, where $2 \leq k \leq n-1$. Take any $S \subseteq N\backslash T, S \neq \emptyset$, and any $j \in T$. From Lemma 7, we have[8]

$$\begin{aligned} d_{N\backslash T}(S,v) &= d_{N\backslash(T\backslash\{j\})}(S,v) + d_{N\backslash(T\backslash\{j\})}(S \cup \{j\}, v) \\ &= \sum_{R \in 2^{T\backslash\{j\}}} d_N(S \cup R, v) + \sum_{R \in 2^{T\backslash\{j\}}} d_N(S \cup R \cup \{j\}, v) \\ &= \sum_{R \in 2^T} d_N(S \cup R, v), \end{aligned}$$

where the second equality holds from the induction hypothesis, which completes the proof. □

The main theorem of this section is given as follows. This theorem says that we can calculate the Shapley value of any subgame by using the coefficients. We skip the proof since it is obvious from Lemma 8 and the fact that the Shapley value coincides with the coefficient.

[8]In the statement of Lemma 7, replace N with $N\backslash(T\backslash\{j\})$. Then, the next equation can be obtained.

Theorem 7 *Let (N, v) be a game and let $T \subseteq N, T \neq \emptyset$. Then, we have*

$$\phi_i(T, v) = \sum_{R \in 2^{N \setminus T}} d_N(\{i\} \cup R, v),$$

for all $i \in T$.

Now, we interpret the results of this section. First, we show that Lemma 7 induces the Balanced Contribution Property, introduced by Myerson (1980). To see this, take any $i, j \in N$. From Lemma 7, we have

$$\begin{aligned} d_{N\setminus\{j\}}(\{i\}, v) &= d_N(\{i\}, v) + d_N(\{i, j\}, v), \\ d_{N\setminus\{i\}}(\{j\}, v) &= d_N(\{j\}, v) + d_N(\{i, j\}, v). \end{aligned}$$

By applying the fact that the coefficients coincide with the Shapley value, we have

$$\begin{aligned} \phi_i(N\setminus\{j\}, v) &= \phi_i(N, v) + d_N(\{i, j\}, v), \\ \phi_j(N\setminus\{i\}, v) &= \phi_j(N, v) + d_N(\{i, j\}, v). \end{aligned}$$

It follows that

$$\phi_i(N, v) - \phi_i(N\setminus\{j\}, v) = \phi_j(N, v) - \phi_j(N\setminus\{i\}, v),$$

which is equivalent to the statement of Balanced Contribution Property. The point is that the equal value, $-d_N(\{i, j\}, v)$, appears in the coefficient.

Namely, Lemma 7 can be interpreted as a variation of Balanced Contribution Property. We now extend the interpretation by using Theorem 7.

Let (N, v) be a game and take any coalition $S \subseteq N, S \neq \emptyset$, any player $i \in S$. Suppose that, at first, players in S were bargaining over the division of $v(S)$. The Shapley value was employed as a division rule, and i got $\phi_i(S, v)$. Now assume that players in $N\setminus S$ merged into the bargaining. For player i, the difference of the Shapley value, $\phi_i(S, v) - \phi_i(N, v)$, can be interpreted as the surplus by the cooperation of $N\setminus S$. By using Theorem 7, the difference can be rewritten as follows.

$$\begin{aligned} \phi_i(N, v) - \phi_i(S, v) &= d_N(\{i\}, v) - \sum_{R \in 2^{N\setminus S}} d_N(\{i\} \cup R, v) \\ &= -\sum_{R \in 2^{N\setminus S}: R \neq \emptyset} d_N(\{i\} \cup R, v). \end{aligned}$$

It is intuitively natural to expect that the change in the payoff of i will be affected by the relationship between i and coalitions in $N\setminus S$. From the equation above, we can interpret that the coefficients, $d_N(\{i\}\cup R, v)$, measure the relationship in a numerical way. The reason is that, if we measure the values $d_N(\{i\} \cup R, v)$ by multiplying -1 and take the sum, then the total change in the payoff can be obtained. In other words, the cooperative surplus for player i can be calculated.

The results in this section are remarkable in that mathematically simple values, coefficients, can not only calculate the Shapley value of all players of all subgames, but also propose an important interpretation.

6 Concluding Remarks

We can extend the discussion in this paper to the weighted Shapley value. For any positive weight $\omega \in \mathbb{R}^n$, $\omega_i > 0, i = 1, \cdots n$, let $\bar{u}^\omega_S \in \Gamma^N$ denote the following game.

$$\bar{u}^\omega_S(T) = \begin{cases} \omega_i, i \in S \cap T & \text{if } |S \cap T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the following theorem holds.

Theorem 8 (Theorem 4 of Yokote (2013)) *Let ω be any positive weight. Then, the set of games $\{u_{\{i\}} : i \in N\} \cup \{\bar{u}^\omega_S : |S| \geq 2\}$ is a linear basis of Γ^N. Moreover, when we express a game by a linear combination of the linear basis, the coefficient of $u_{\{i\}}$ is equal to the weighted Shapley value with positive weight ω of $i \in N$.*

The point is that the weighted Shapley value with positive weight ω prescribes $\mathbf{0}$ vector to $\bar{u}^\omega_S, S \subseteq N, |S| \geq 2$. As a result, the null space and the inverse problem can be generalized to the case of the weighted Shapley value.

References

[1] Chun, Y. (1991): "On the symmetric and weighted Shapley values," International Journal of Game Theory, 20, 183-190.

[2] Dragan, I (2005): "On the inverse problem for semivalues of cooperative TU games," International Journal of Pure and Applies Mathematics, 4, 539-555.

[3] Dragan, I (2012): "On the inverse problem for multiweighted Shapley values of cooperative TU games," International Journal of Pure and Applies Mathematics, 75, 279-287.

[4] Dragan, I., J. Potters and S. Tijs (1989): “Superadditivity for solutions of coalitional games,” Libertas Mathematica, 9, 101-110.

[5] Harsanyi, J.C. (1959): “A bargaining model for cooperative n-person games,” In: Tucker AW, Luce RD (ed) Contributions to the theory of games IV. Princeton UP, Princeton, pp 325-355.

[6] Hart, S., and A. Mas-Colell (1989): “Potential, value and consistency,” Econometrica, 57, 589-614.

[7] Kleinberg, N. L., and J. H. Weiss (1985): “Equivalent N-person games and the null space of the Shapley value,” Mathematics of Operations Research, 10, 233-243.

[8] Myerson, R. (1980): “Conference structures and fair allocation rules, ” International Journal of Game Theory, 9, 169-182.

[9] Owen, G. (1972): “Multilinear extensions of games,” Management Sciences, 18, 64-79.

[10] Peleg, B., and P. Sudhölter. (2007): Introduction to the theory of cooperative games, 2nd ed, Springer-Verlag.

[11] van den Brink, R. (2001): “An axiomatization of the Shapley value using a fairness property,” International Journal of Game Theory, 30, 309-319.

[12] Yokote, K. (2013): “Strong Addition Invariance and axiomatization of the weighted Shapley value,” Working Paper.

[13] Young, H.P. (1985): “Monotonic solutions of cooperative games,” International Journal of Game Theory, 14, 65-72.