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Weighted Position Value

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Abstract

This paper provides a non-symmetric generalization of the position value for communication situations. The definition of the weighted position value is motivated by the two different kinds of asymmetry in communication situations. It is shown that the weighted position value is characterized by component efficiency and a modification of balanced link contributions that is used by Slikker (International Journal of Game Theory 2005, Vol. 33, pp.505-514) to characterize the (non-weighted) position value.

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1 Introduction

A communication situation is one in which participants with an economic or social problem obtain a payoff through cooperation, and their cooperation is restricted to the exogenously given communication structure. Two major allocation rules in communication situations assume a symmetric treatment of players. The first is the Myerson value introduced by Myerson (1977). The Myerson value is the *non-weighted* Shapley value applied to a graph-restricted game, which is derived from an original communication situation so as to represent a communication restriction among players. He characterized the Myerson value by component efficiency and fairness. His fairness axiom requires that two end point players of each link receive *equal* damage by deleting the link, and thus, well describes the symmetric treatment of players in the Myerson value. An alternative allocation rule for communication situations is the position value introduced by Borm, Owen, and Tijs (1992). The position value is defined using a link game where each link in the communication situation is viewed as a player. In the defining the position value, the *non-weighted* Shapley value for the link game is calculated first, and then the value of a link is *equally* divided by the two end point players of the link. The sum of half of the Shapley value of the links that a player has is his position value.

In this short paper, we add a non-symmetric flavor into communication situations. We define the weighted position value in communication situations, motivated by two sources of asymmetry. The first is asymmetry among links. Asymmetry among links is embedded in the definition of the weighted position value, by applying the weighted Shapley value (Shapley (1953a)) instead of the Shapley value to the link games. The second is asymmetry among players, described by the unequal division of the value between the two end point players. We axiomatize the weighted position value, by component efficiency and weighted balanced link contributions which is the modification of balanced link contributions introduced by Slikker (2005). In addition, as a result of considering the relationship between two kinds of asymmetry, we obtain a new allocation rule in communication situations. This allocation rule is characterized by the component efficiency and another form of the weighted balanced link contributions, in which asymmetry among players appears in the same manner as the weighted Shapley value and the weighted Myerson value.

There are studies on asymmetry between players in communication situations, but not on asymmetry between links. Haeringer (1999) considered asymmetries between players in communication situations and introduced the weighed Myerson value, which is simply defined by applying the *weighted* Shapley value for graph-restricted games. Slikker and van den Nouweland (2000) considered the weight system by Kalai and Samet (1987), and extended the weighted Myerson value to hierarchical structures among players. In the case of the position value, Kongo (2007) considered an extension of the position value to non-symmetric situation, but his extension was only for asymmetry among players.

This paper is organized as follows: In the next section, the basic notations and definitions used in this paper are given. The weighted position value is provided in Section 3 and its axiomatic characterization is presented in Section 4. In Section 5, the relationship between two kinds of asymmetry is considered. Section 6 concludes the paper.

2 Notations and definitions

Let N be a set of finite players and $|N| = n$ where $|\cdot|$ represents the cardinality of the set. For $S \subseteq N$, let $N - S$ denote a set $N \setminus S$. A function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a *cooperative game with transferable utility*, or simply, a *game*. A set of all games on N is denoted by $\mathscr{V}^N$. If $v \in \mathscr{V}^N$ satisfies $v(\{i\}) = 0$ for any $i \in N$, v is called a *zero-normalized* game. A set of all zero-normalized games on N is denoted by $\mathscr{V}_0^N$.

For any $S \subseteq N$, let $u_S \in \mathscr{V}^N$ be a *S-unanimity game* defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases},$$

for any $T \subseteq N$. It is well known that any $v \in \mathscr{V}^N$ is represented as a linear combination of unanimity games, that is,

$$v = \sum_{S \subseteq N; S \neq \emptyset} \Delta_v(S) u_S,$$

where $\Delta_v(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$ is a *dividend* of S.

A solution for a game $v \in \mathscr{V}^N$ is a function which associates a game v with a payoff vector $x = (x_i)_{i \in N}$ that satisfies $\sum_{i \in N} x_i \leqq v(N)$. A well-known solution is the *Shapley value* ϕ (Shapley (1953b)). It is defined by

$$\phi_i(v) = \sum_{S \subseteq N; S \ni i} \frac{\Delta_v(S)}{|S|},$$

for any $i \in N$.

A non-symmetric generalization of the Shapley value is considered by Shapley (1953a). For any player i, let $\theta_i > 0$ denote a positive weight for i and $\theta = (\theta_i)_{i \in N}$ denote a positive weight vector of players. Given a weight vector θ, the "weighted" version of the Shapley value is defined as follows. For any $i \in N$,

$$\phi_i^\theta(v) = \sum_{S \subseteq N; S \ni i} \Delta_v(S) \frac{\theta_i}{\sum_{j \in S} \theta_j}.$$

ϕ^θ is called the *weighed Shapley value*.

Next, we consider communication relations between players. Given a player set N, the bilateral communication channels between the players in N are described by an undirected graph $g \subseteq g^* = \{\{i,j\} | i,j \in N, i \neq j\}$. A set of all graphs on N is denoted by G^N. Each communication channel in a graph is called a *link*, and it is represented as ij or ℓ instead of $\{i,j\}$. For $g \subseteq g^*$ and $i \in N$, $g_i = \{\ell \in g | i \in \ell\}$ is a set of links that player i belongs to. Given a graph g, if there exists a finite sequence of players $i_1, \ldots, i_H$ such that $i_1 = i, i_H = j$ and $i_h i_{h+1} \in g$ for any $h = 1, \ldots, H-1$, then i *is connected to* j in the graph. Given a graph g, players can communicate with each other, if and only if they are connected to each other in the graph g. Let

$$N/g = \{\{j \in N | i \text{ is connected to } j \text{ in } g\} \cup \{i\} \mid i \in N\}.$$

N/g represents a collection of a set of communicable players. For any $S \subseteq N$, let $g(S) = \{\ell \in g | \ell \subseteq S\}$ denote a restriction of g on S. By $g(S)$, S/g is defined in the same manner as N/g, that is,

$$S/g = \{\{j \in S | i \text{ is connected to } j \text{ in } g(S)\} \cup \{i\} \mid i \in S\}.$$

Fix player set N. Let $v \in \mathcal{V}_0^N$ and $g \in G^N$. A pair (v,g) is called a *communication situation*. An allocation rule for communication situations is a n-dimensional vector valued function on $\mathcal{V}_0^N \times G^N$. Let ψ be an allocation rule for communication situations. The Myerson value (Myerson (1977)) is a famous allocation rule and is extended to the case with players' asymmetry by Haeringer (1999). Let θ be a weight vector for players in N. The *weighted Myerson value* μ^θ is defined as follows. For any $i \in N$,

$$\mu_i^\theta(v,g) = \phi_i^\theta(v^g),$$

where $v^g \in \mathcal{V}^N$ is a graph-restricted game by g defined by $v^g(S) = \sum_{C \in S/g} v(C)$ for any $S \subseteq N$. Thus, the weighted Myerson value is the weighted Shapley value, applied to the graph-restricted game.

Another allocation rule in communication situations is the position value. Given a communication situation (v,g), a *link game* $r \in \mathcal{V}^g$ of the communication situation is defined by

$$r(g') = v^{g'}(N),$$

for any $g' \subseteq g$. The position value π for a communication situation (v,g) is defined as follows. For any player $i \in N$,

$$\pi_i(v,g) = \sum_{\ell \in g_i} \frac{1}{2} \phi_\ell(r).$$

Thus, the position value of the players is such that first the allocation of each of the links in the link game is measured by the Shapley value; the players who are at the end point of a link and implicitly assumed to have equal bargaining power split the allocation of the link equally, and the player collects that amount from all the links to which he belongs.

An axiomatic foundation for the position value was first presented by Borm, Owen, and Tijs (1992). Their characterization works in the restricted class of *cycle-free* graphs. Later, Slikker (2005) found the way to characterize the position value for an unrestricted class of communication situations through the following two axioms. Let ψ denote an allocation rule for communication situations. Given a communication situation (v,g),

Component Efficiency: For any component $C \in N/g$,

$$\sum_{i \in C} \psi_i(v,g) = v(C).$$

Balanced link contributions: For any $i \in N$ and any $j \in N$,

$$\sum_{\ell \in g_j} \left(\psi_i(v,g) - \psi_i(v,g-\ell)\right) = \sum_{\ell' \in g_i} \left(\psi_j(v,g) - \psi_j(v,g-\ell')\right).$$

Component efficiency is quite natural for communication situations. This property requires that only the players who can communicate with each other can cooperate, and they divide the worth of their cooperation among themselves. Balanced link contributions, is a concept related to the balance between any two players' bargaining power and thus the fair treatment accorded to the two players in terms of their links. It requires that the sum of the differences of the payoff for player i in severing link ℓ over all the links to which player j belongs, is equal the sum of the differences of the payoff for player j in severing link ℓ' over all the links to which player i belongs.

Taking the viewpoint that the "average" of the differences of the payoff instead of the "total sum" should be balanced, balanced link contributions is revised to the following *average balanced link contributions*:

Average balanced link contributions: For any $i \in N$ and any $j \in N$,

$$\frac{1}{|g_j|} \sum_{\ell \in g_j} (\psi_i(v,g) - \psi_i(v,g-\ell)) = \frac{1}{|g_i|} \sum_{\ell' \in g_i} \left(\psi_j(v,g) - \psi_j(v,g-\ell')\right).$$

One advantage of average balanced link contributions over balanced link contributions is its simplicity. The average of the differences of the player's payoff over the links of his opponent is considered as an expected differences of his payoff when the opponent cuts one of his links at random. Thus, the average balanced link contributions suggests that the threat of player i to j, which is measured by the expected difference of player j's payoff when i cut one of his link at random, balances with the threat of player j to i.

In the literature on communication situations, it remains open as to what kinds of allocation rules satisfy the average balanced link contributions. Our attempt to extend the position value to non-symmetric cases in two directions gives some insight into the form of such an allocation rule. This point is considered in Section 5.

3 Weighted position value

One step of extending the position value to non-symmetric situations is considered by Kongo (2007). He dropped a symmetric treatment among players in order to respond to economic and social situations where players in an underlying game should be treated in an asymmetric way, because of personal characteristics that are not reflected in the game itself (e.g., they have the different bargaining abilities or input different effort levels to achieve or maintain the underlying situation).

Let $\theta \in \mathbb{R}^N_{++}$ be a positive weight vector for players. Then, the *player-wise weighted position value* π^θ for (v,g) is defined by

$$\pi_i^\theta(v,g) = \sum_{ih \in g_i} \frac{\theta_i}{\theta_i + \theta_h} \phi_{ih}(r),$$

for any $i \in N$. Thus, the symmetric assumption between players in the position value, which is expressed by the equal split of the allocation of a link between players at the end-point of the link, is now replaced by an asymmetry between the two players in a such way that the allocation of the link is divided proportionally to their weights.

Another way of extending the position value to a non-symmetric situation, is to consider an asymmetry among links. The asymmetry among links in underlying game is often caused by the physical aspects. One typical factor is the length of each link, which may influence the speed of communication between the players and the accuracy of the information conveyed on the link. Other factor is the different construction costs for the links, which may be sunk in the current situation, but for which it may be necessary to compensate from the profits of the underlying game.

Let $\sigma \in \mathbb{R}^{g^*}_{++}$ be a positive weight vector for links. Here, for notational convenience, we define the weights for the potential links that may not appear in the current communication situation (v,g). Then, the *link-wise weighted position value* π^σ is defined by

$$\pi_i^\sigma(v,g) = \sum_{ih \in g_i} \frac{1}{2} \phi_{ih}^\sigma(r),$$

for any $i \in N$. Thus, the asymmetry among links is now represented in the weighted Shapley value for the link game.

It is quite natural to consider the mixture of both kinds of asymmetric extensions of position value. Let $\theta \in \mathbb{R}_{++}^N$ and $\sigma \in \mathbb{R}_{++}^{g^*}$ denote weights on players in N and links in g^* respectively. A pair $w = (\theta, \sigma)$ is called a *weight structure* for communication situations.

Definition 1. *Given a weight structure $w = (\theta, \sigma)$ for communication situations, the weighted position value π^w for (v,g) is defined as follows. For any player $i \in N$,*

$$\pi_i^w(v,g) = \sum_{ih \in g_i} \frac{\theta_i}{\theta_i + \theta_h} \phi_{ih}^\sigma(r).$$

4 Axiomatic characterization

Given a weight structure $w = (\theta, \sigma)$, the balanced link contributions is modified for asymmetric cases as follows:

Weighted balanced link contributions: For any $i \in N$ and any $j \in N$,

$$\sum_{jk \in g_j} \sigma_{jk} \frac{\theta_j}{\theta_j + \theta_k} \left(\psi_i(v,g) - \psi_i(v, g - jk)\right) = \sum_{ih \in g_i} \sigma_{ih} \frac{\theta_i}{\theta_i + \theta_h} \left(\psi_j(v,g) - \psi_j(v, g - ih)\right).$$

Thus, balanced link contributions is extended in the two directions. First, an asymmetry among players is reflected in the proportion $\frac{\theta_j}{\theta_j + \theta_k}$ of the difference of player i's payoff, when player j breaks down his link jk. Second, an asymmetry among links is reflected in the multiplier σ_{jk} for the difference of the player i's payoff in player j's serving link jk.

The next theorem characterizes the weighted position value. Since when $\sigma_\ell = \sigma_{\ell'}$ holds for any two links ℓ and ℓ', weighted balanced link contributions coincides with the one considered in Kongo (2007), which he used to characterize the player-wise position value. Thus, the next theorem includes Theorem 1 in Kongo (2007).

Theorem 1. *Given a weight structure $w = (\theta, \sigma)$, there exists a unique allocation rule on communication situations that satisfies component efficiency and weighted balanced link contributions. This allocation rule is the weighted position value π^w.*

Proof. Let a weight structure $w = (\theta, \sigma)$ be given. The following proof is modification of the proof of Lemma 3.1 and Theorem 3.1 of Slikker (2005).

First, we show that π^w satisfies the two axioms. For the component efficiency, for any $C \in N/g$,

$$\sum_{k\in C}\pi_k^w(v,g) = \sum_{ij\in g(C)}\Big(\frac{\theta_i}{\theta_i+\theta_j}+\frac{\theta_j}{\theta_i+\theta_j}\Big)\phi_{ij}^{\sigma}(r) = \sum_{ij\in g(C)}\phi_{ij}^{\sigma}(r)$$
$$= \sum_{ij\in g(C)}\phi_{ij}^{\sigma}(r|_{g(C)}) = r(g(C)) = v(C),$$

where $r|_{g(C)} \in \mathscr{V}^{g(C)}$ is a restriction of $r \in \mathscr{V}^g$ on $g(C)$. In the above equation, the third equality holds because for any $ij \in g(C)$ and any $g' \subseteq g - ij$, the marginal contributions of ij to g' are equal to those of ij to $g' \cap g(C)$. The fourth equality is by the efficiency of the weighted Shapley value.

For any $g' \subseteq g$, let $\sigma(g')$ denote $\sum_{\ell\in g'}\sigma_\ell$. If a domain for the summation is empty, we put the value for the summation at zero.

For weighted balanced link contributions, for any $i, j \in N$,

$$\sum_{jk\in g_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}(\pi_i^w(v,g)-\pi_i^w(v,g-jk))$$
$$= \sum_{jk\in g_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}\Big(\sum_{ih\in g_i}\frac{\theta_i}{\theta_i+\theta_h}\sum_{\substack{g'\subseteq g\\ g'\ni ih}}\sigma_{ih}\frac{\Delta_r(g')}{\sigma(g')}$$
$$-\sum_{ih\in(g-jk)_i}\frac{\theta_i}{\theta_i+\theta_h}\sum_{\substack{g'\subseteq g-jk\\ g'\ni ih}}\sigma_{ih}\frac{\Delta_{r|_{g-jk}}(g')}{\sigma(g')}\Big)$$
$$= \sum_{jk\in g_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}\Big(\sum_{g'\subseteq g}\frac{\Delta_r(g')}{\sigma(g')}\sum_{ih\in g'_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}$$
$$-\sum_{g'\subseteq g-jk}\frac{\Delta_{r|_{g-jk}}(g')}{\sigma(g')}\sum_{ih\in g'_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}\Big)$$
$$= \sum_{jk\in g_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}\sum_{\substack{g'\subseteq g\\ g'\ni jk}}\frac{\Delta_r(g')}{\sigma(g')}\sum_{ih\in g'_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}$$
$$= \sum_{g'\subseteq g}\sum_{jk\in g'_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}\frac{\Delta_r(g')}{\sigma(g')}\sum_{ih\in g'_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}$$
$$= \sum_{g'\subseteq g}\frac{\Delta_r(g')}{\sigma(g')}\sum_{ih\in g'_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}\sum_{jk\in g'_j}\sigma_{jk}\frac{\theta_j}{\theta_j+\theta_k}$$
$$= \sum_{ih\in g_i}\sigma_{ih}\frac{\theta_i}{\theta_i+\theta_h}(\pi_j^w(v,g)-\pi_j^w(v,g-ih)),$$

where the third equality is by the fact that for any $g' \subseteq g - jk$ $\Delta_r(g') = \Delta_{r|_{g-jk}}(g')$ and the last equality follows by the same arguments as for the first four equalities.

To prove the uniqueness, let ψ satisfy the two axioms. The proof is by induction of the number of links in g. If $g = \emptyset$, component efficiency implies $\psi(v,g) = v(\{i\}) = \pi^w(v,g)$ for any $i \in N$ thus, $\psi = \pi^w$. Let $m \geq 1$. Suppose that $\psi = \pi^w$ holds for any graph that contains less than $m-1$ links and consider the case that g contains m

links. Fix $C \in N/g$. If C is singleton, component efficiency implies $\psi(v,g) = v(\{i\}) = \pi^w(v,g)$ for $i \in C$. If $|C| \geq 2$, without loss of generality, let $C = \{1,2,\ldots,c\}$. Applying weighted balanced link contributions to pairs $\{1,2\},\{1,3\},\ldots,\{1,c\}$, we obtain

$$\begin{aligned}
&\sum_{2k \in g_2} \sigma_{2k} \frac{\theta_2}{\theta_2+\theta_k} \psi_1(v,g) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \psi_2(v,g) \\
&\quad = \sum_{2k \in g_2} \sigma_{2k} \frac{\theta_2}{\theta_2+\theta_k} \psi_1(v,g-2k) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \psi_2(v,g-1h) \\
&\qquad = \sum_{2k \in g_2} \sigma_{2k} \frac{\theta_2}{\theta_2+\theta_k} \pi_1^w(v,g-2k) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \pi_2^w(v,g-1h);
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\sum_{ck \in g_c} \sigma_{ck} \frac{\theta_c}{\theta_c+\theta_k} \psi_1(v,g) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \psi_c(v,g) \\
&\quad = \sum_{ck \in g_c} \sigma_{ck} \frac{\theta_c}{\theta_c+\theta_k} \psi_1(v,g-ck) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \psi_c(v,g-1h) \\
&\qquad = \sum_{ck \in g_c} \sigma_{ck} \frac{\theta_c}{\theta_c+\theta_k} \pi_1^w(v,g-ck) - \sum_{1h \in g_1} \sigma_{1h} \frac{\theta_1}{\theta_1+\theta_h} \pi_c^w(v,g-1h);
\end{aligned}$$

Also, by component efficiency,

$$\sum_{i \in C} \psi_i(v,g) = v(C).$$

These c equalities form a regular system of linear equations in c variables and it has a unique solution which is a weighted position value. Hence, for any $i \in C$, ψ_i coincides with π_i^w. For any $i \in C' \in N/g$ we can prove the coincidence between ψ_i and π_i^w in the same way. By induction of m, the proof is completed. □

Corollary 1 (Kongo (2007))**.** *The player-wise weighted position value π^θ is characterized by the component efficiency and the player-wise weighted balanced link contributions defined by: for any $i \in N$ and for any $j \in N$,*

$$\sum_{jk \in g_j} \frac{\theta_j}{\theta_j+\theta_k} \left(\psi_i(v,g) - \psi_i(v,g-jk)\right) = \sum_{ih \in g_i} \frac{\theta_i}{\theta_i+\theta_h} \left(\psi_j(v,g) - \psi_j(v,g-ih)\right).$$

Corollary 2. *The link-wise weighted position value π^σ is characterized by the component efficiency and the link-wise weighted balanced link contributions defined by: for any $i \in N$ and for any $j \in N$,*

$$\sum_{jk \in g_j} \sigma_{jk} \left(\psi_i(v,g) - \psi_i(v,g-jk)\right) = \sum_{ih \in g_i} \sigma_{ih} \left(\psi_j(v,g) - \psi_j(v,g-ih)\right).$$

5 Basis for the weights

Let us consider the relationship between weights for players and weights for links. Given a weight vector θ for players, if we adapt the viewpoint that the weight of a link is determined by the weights for the end point players for the link, a candidate of σ_{ih} would be

$$\sigma^*_{ih}(\theta) = \theta_i + \theta_h.$$

One may suppose that average of the two players' weights is preferable for the weight of the link, as compared to the sum of the weights of the end point players. Generally speaking, the choice of the appropriate form of the weight is quite a difficult question and it strongly depends on the context. Fortunately, however, this point does not matter because the scale of the weight vector for players is independent of that of the weight for links, thanks to the separability of the two sources of the weight. Both the average and the total sum generate the same result.

Let $\sigma^*(\theta) = (\sigma^*_\ell(\theta))_{\ell \in g^*}$. For weight structure $w^*(\theta) = (\theta, \sigma^*(\theta))$, the weighted balanced contributions is reduced to the following: For any $i \in N$ and for any $j \in N$,

$$\sum_{jk \in g_j} \theta_j \left(\psi_i(v,g) - \psi_i(v,g-jk)\right) = \sum_{ih \in g_i} \theta_i \left(\psi_j(v,g) - \psi_j(v,g-ih)\right).$$

Dividing the above equation by $\theta_i \theta_j$, we obtain the following property.

Class-balanced link contributions: For any $i \in N$ and any $j \in N$,

$$\sum_{jk \in g_j} \frac{1}{\theta_i} \left(\psi_i(v,g) - \psi_i(v,g-jk)\right) = \sum_{ih \in g_i} \frac{1}{\theta_j} \left(\psi_j(v,g) - \psi_j(v,g-ih)\right).$$

Class-balanced contributions requires that the sum of link contributions normalized by player's weight θ_i should balance.

The "player" version of the class balanced link contributions is defined as follows:

Class-balanced contributions: For any $i \in N$ and any $j \in N$,

$$\frac{1}{\theta_i} \left(\psi_i(v,g) - \psi_i(v,g-g_j)\right) = \frac{1}{\theta_j} \left(\psi_j(v,g) - \psi_j(v,g-g_i)\right).$$

Thus, while in class-balanced link contributions the sum of the difference of player i's payoff, when j holds all the links and when j cuts one of his links over all the links of player j is divided by the weight of player i, in class-balanced contributions the difference of the payoff of player i when his opponent j holds all the links and when j cuts his all links is divided by the weight of player i.

Now, we present the following theorems, which axiomatize allocation rules satisfying class-balanced contributions and class-balanced link contributions respectively. Theorem 2 is by Slikker and van den Nouweland (2000), and Theorem 3 is an immediate consequence of the above argument and Theorem 1.

Theorem 2 (Slikker and van den Nouweland (2000))**.** *Given a weight vector θ for players, there is a unique allocation rule on communication situations, that satisfies component efficiency and class-balanced contributions. This is the weighed Myerson value μ^θ.*

Theorem 3. *Given a weight vector θ for players, there is a unique allocation rule on communication situations that satisfies component efficiency and class-balanced link contributions. This solution is defined by*

$$\lambda_i^{\theta}(v,g) = \pi_i^{w^*(\theta)}(v,g),$$

for any $i \in N$.

Recall that λ^{θ} and class-balanced link contributions are the special case of the weighed position value π^w and weighted balanced link contributions. Since Theorem 1 works for any given weight structure w, the proof of Theorem 3 is obvious from the definition of λ^{θ} and the weight structure $w^*(\theta)$.

In the above discussion, we consider that the weight for links is determined by the weight for the players. Next, we consider that the weight for players is determined by the structure of the links. Here, there is an appropriate source of the weight for the players in terms of the links: the number of the links that each player has. In the co-author model (Jackson and Wolinsky (1996)), it is assumed that a player equally allocated his constant effort capacity to all the links that he has, and thus his per capita effort towards each link decreases as the total number of his links increases. Similarly, we consider that the weight for the player is represented as his per capita effort. Then, the weight of player i is

$$\theta_i^*(g) = \frac{1}{|g_i|},$$

when $g_i \neq \emptyset$ and $\theta_i^*(g)$ equals an arbitrary positive number $a > 0$ when $g_i = \emptyset$.[1]

Now we obtain the way that both weights for players and links are endogenously determined by the underlying communication situation itself. Given (v,g), first the weight for players is determined as $\theta^*(g)$ and then the weight for the links is $\sigma^*(\theta^*(g))$. By using this weight structure $w^{**}(g) = (\theta^*(g), \sigma^*(\theta^*(g)))$ which depends on the current communication situation, we have a new allocation rule for communication situation.

Definition 2. *An allocation rule κ for communication situations is defined as follows. For each (v,g) and $i \in N$,*

$$\kappa_i(v,g) = \pi_i^{w^{**}(g)}(v,g) = \sum_{ih \in g_i} \frac{\theta_i^*(g)}{\theta_i^*(g) + \theta_h^*(g)} \phi_{ih}^{\sigma^*(\theta^*(g))}(r).$$

Let us consider an allocation rule that satisfies average balanced link contributions. Class-balanced link contributions can be reduced to:

$$\theta_j \sum_{jk \in g_j} (\psi_i(v,g) - \psi_i(v,g-jk)) = \theta_i \sum_{ih \in g_i} (\psi_j(v,g) - \psi_j(v,g-ih)),$$

for any $i \in N$ and any $j \in N$. Fix communication situation (v,g) and $\hat{\theta}_i = \frac{1}{|g_i|}$ for $i \in N$. An allocation rule $\lambda^{\hat{\theta}}$ satisfies the above equation for communication situation (v,g). However, this allocation rule $\lambda^{\hat{\theta}}$ does not satisfy average balanced link contributions because, in general, $\lambda^{\hat{\theta}}$ does not satisfy the above condition for other communication

[1] Note that if $g_i = \emptyset$, his weight is *not* determined by the structure of the links, since, in our setting, the weight must be positive. However, if we consider component efficiency together, the player who has no links always obtains nothing and thus the weight of the player has no meaning.

situations (v', g'). On the other hand, the allocation rule κ does not satisfy the average balanced link contributions, either. This is because while $\kappa(v,g) = \lambda^{\hat{\theta}}(v,g)$ holds, $\kappa(v,g')$ does not coincide with $\lambda^{\hat{\theta}}(v,g')$ for $g' \subsetneq g$. Thus, both exogenous and endogenous approaches can not answer the form of the solution that satisfies the average balanced link contributions.

6 Conclusions

This paper, extends the position value to communication situations with asymmetry among players, as well as among links. The weighted position value is defined, with respect to weight structure $w = (\theta, \sigma)$. We axiomatized the weighted position value by the component efficiency and weighted balanced link contributions.

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