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Nonsatiation and existence of competitive equilibrium

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Abstract

In this paper, we provide a new assumption on satiation of preferences that is weaker than the standard nonsatiation assumption and Allouch and Le Van's (2008a; 2008b) weak nonsatiation. It allows, under certain conditions, preferences to be satiated only inside the individually rational feasible consumption sets. Moreover, just like the two nonsatiation assumptions, our assumption depends solely on the characteristics of consumers.

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Key words: Satiation, Quasi-equilibrium, Individually rational feasible consumption.

1. Introduction

Insatiability of consumers' preferences is a standard assumption in the classical general equilibrium theory (Arrow and Debreu, 1954; Debreu, 1959, among others). This assumption, in its strong form, asserts that consumers' preferences are insatiable over the entire consumption set. However, in some cases, we observe that consumption sets are naturally compact (see Mas-Colell, 1992) and every continuous preference has therefore at least one satiation point. As is well known in the literature, a simple way to avoid this inconsistency is to assume that when a preference has satiation points, they

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are always outside the individually rational feasible consumption set. [1] This weaker version of the standard nonsatiation allows preferences to be satiated, but excludes the case in which satiation occurs inside the individually rational feasible consumption sets.

It has been known that a competitive equilibrium may fail to exist when preferences are satiated in the individually rational feasible consumption set. Recently, however, Allouch and Le Van (2008a,b) have shown that even if there exists a consumer whose preference has satiation points in his or her individually rational feasible consumption set, one can still obtain the existence of competitive equilibrium by assuming that the preference also has at least one satiation point outside the set. The assumption is a direct generalization of the standard nonsatiation assumption, and therefore, they call it "weak nonsatiation".

Won and Yannelis (2006) introduce a different assumption that allows for satiation inside the individually rational feasible consumption sets. Their existence results are quite general. For example, in their proofs, individually rational feasible consumption sets do not need to be bounded and consumers' preferences are allowed to be non-ordered. Moreover, Won and Yannelis's results apply to the case in which satiation occurs only insides the individually rational feasible consumption sets, while Allouch and Le Van's (2008a,b) result does not. In fact, Won and Yannelis's assumption contains Allouch and Le Van's weak nonsatiation as a special case. However, it is worth noting that while Won and Yannelis's assumption contains a restriction on the price system, weak nonsatiation depends solely on the characteristics of consumers just like the standard nonsatiation.

The main contribution of this paper is to establish the existence of competitive equilibrium by introducing a new assumption that is weaker than Allouch and Le Van's (2008a,b) weak nonsatiation and therefore the standard nonsatiation assumption. Our assumption allows each consumer's preference to be satiated only inside the individually rational feasible consumption set, provided that at least one satiation point lies on a "boundary" of the set. Roughly speaking, the "boundary" of one's individually rational feasible consumption set is defined as the set of individually rational feasible consump-

[1]A consumption bundle is said to be individually rational feasible if it can be achieved by a trade in which every consumer involved attains at least the same utility as that gained from his or her initial endowment. For the existence proof under this assumption, see Bergstrom (1976); Dana and Le Van (1999), for example.

tion bundles beside each of which there are another consumption bundles that are at least as good as the initial endowment for the consumer but are not individually rational feasible. Although our existence results, unlike Won and Yannelis (2006), rely on the boundedness of individually rational feasible consumption sets and the existence of ordered preferences, our nonsatiation assumption does not imply Won and Yannelis's assumption. Moreover, just like Allouch and Le Van's weak nonsatiation and standard nonsatiation, our assumption depends solely on the characteristics of consumers.

This paper is organized as follows. In Section 2, we describe the model and then introduce the new assumption. In Section 3, we provide our main results. In Section 4, we consider an alternative to the new assumption and provide some related results. As a concluding remark, in Section 5, we compare our assumption with the assumption introduced by Won and Yannelis (2006). Some of the proofs of propositions in Section 3 are provided in the Appendix.

2. Model and Assumptions

2.1. Model

We consider a pure exchange economy $\mathcal{E}$ with ℓ commodities and n consumers ($\ell, n \in \mathbb{N}$). [2] For convenience, let I be the set of all consumers, that is, $I = \{1, \cdots, n\}$. Each consumer $i \in I$ is characterized by a consumption set $X_i \subset \mathbb{R}^\ell$, an initial endowment $\omega_i \in \mathbb{R}^\ell$, and a utility function $u_i : X_i \to \mathbb{R}$. Let $X = \prod_{i \in I} X_i$ with a generic element $x = (x_i)_{i \in I}$, and put $\omega = (\omega_i)_{i \in I} \in \mathbb{R}^{\ell n}$.

The pure exchange economy $\mathcal{E}$ is thus summarized by the list

$$\mathcal{E} = \Big(\mathbb{R}^\ell, (X_i, u_i, \omega_i)_{i \in I}\Big).$$

An allocation $x \in X$ is *feasible* if $\sum_{i \in I} x_i = \sum_{i \in I} \omega_i$. Note that we do not allow free disposal. We denote the set of all feasible allocations by F. Let F_i

[2] We use the following mathematical notations. The symbols $\mathbb{N}$, $\mathbb{R}^\ell$ and $\mathbb{R}^\ell_+$ denote the set of natural numbers, the ℓ-dimensional Euclidean space and the nonnegative orthant of $\mathbb{R}^\ell$, respectively. For $x, y \in \mathbb{R}^\ell$, we denote by $x \cdot y = \sum_{j=1}^{\ell} x_j y_j$ the inner product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm. Let $B(x_0, r) = \{x \in \mathbb{R}^\ell : \|x - x_0\| < r\}$ denote the open ball centered at x_0 with radius r. For $a \in \mathbb{R} = \mathbb{R}^1$, we denote by $|a|$ the absolute value of a. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote by (a, b) and $[a, b]$, the open interval and the closed interval between a and b, respectively. For a set $A \subset \mathbb{R}^\ell$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$, the interior, the closure and the boundary of A in $\mathbb{R}^\ell$, respectively.

be the projection of F onto X_i, and call it *individually feasible consumption set of consumer* $i \in I$. Then, it is easy to check that $F_i = X_i \cap (-\sum_{j \neq i} X_j + \sum_{k \in I}\{\omega_k\})$ for all $i \in I$.

An allocation $x \in X$ is *individually rational feasible* if $x \in F$ and $u_i(x_i) \geq u_i(\omega_i)$ for all $i \in I$. We denote the set of all individually rational feasible allocations by A. Let A_i be the projection of A onto X_i, and call it *individually rational feasible consumption set of consumer* $i \in I$.

Let $R_i = \{x_i \in X_i : u_i(x_i) \geq u_i(\omega_i)\}$ for each $i \in I$. Then, it is easy to check that

$$A_i = R_i \cap \left(-\sum_{j \neq i} R_j + \sum_{k \in I}\{\omega_k\} \right) \quad \text{for all} \quad i \in I.$$

For simplicity of notation, we put

$$G_i = -\sum_{j \neq i} R_j + \sum_{k \in I}\{\omega_k\} \quad \text{for each} \quad i \in I.$$

Note that $A_i = R_i \cap G_i \subset R_i$ for all $i \in I$.

The utility function u_i is *satiated at* $s_i \in X_i$ if s_i maximizes u_i over X_i, and we call the consumption bundle s_i a *satiation point of* u_i. Let S_i denote the set of all satiation points of u_i, that is,

$$S_i = \{s_i \in X_i : u_i(s_i) \geq u_i(x_i) \text{ for all } x_i \in X_i\}.$$

Put $S = \prod_{i \in I} S_i$.

We adopt the following standard definitions of competitive equilibrium and quasi-equilibrium.

Definition 1. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *competitive equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot \omega_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then, $\overline{p} \cdot x_i > \overline{p} \cdot \omega_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} \omega_i$.

Definition 2. An element $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a *quasi-equilibrium of the economy* $\mathcal{E}$ if

(a) for all $i \in I$,

(a-1) $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot \omega_i$,

(a-2) if $u_i(x_i) > u_i(\overline{x}_i)$, then, $\overline{p} \cdot x_i \geq \overline{p} \cdot \omega_i$,

(b) $\sum_{i \in I} \overline{x}_i = \sum_{i \in I} \omega_i$.

2.2. Assumptions

We first make the following two sets of assumptions on the economy $\mathcal{E}$.

Assumption 1. For each $i \in I$,

(a) X_i is closed and convex, (b) X_i is bounded, (c) $\omega_i \in X_i$.

Assumption 2. For each $i \in I$,

(a) u_i is upper semicontinuous on X_i, [3]

(b) u_i is strictly quasi-concave. [4]

The existence of a quasi-equilibrium is ensured, as shown in 3.1, under Assumptions 1 and 2 and our new assumption on satiation of preferences introduced below. To prove the existence of a competitive equilibrium, however, we need some additional assumptions (see 3.2). It is worth noting that in the main existence theorems of this paper (Theorem 2 and 3), Assumption 1 (b) can be weakened to the boundedness of A by the standard truncation technique.

It is easy to check that under Assumptions 1 and 2, we have the following facts.

Fact 1. $S_i \neq \varnothing$ for each $i \in I$.

Fact 2. R_i, G_i and A_i are nonempty, compact and convex in $\mathbb{R}^\ell$ for each $i \in I$.

Especially, the convexity of R_i in Fact 2 follows from the quasi-concavity of u_i, [5] which is implied by Assumptions 1 (a), 2 (a) and 2 (b).

[3] A function $f : X \to \mathbb{R}$ is upper semicontinuous on $X \subset \mathbb{R}^\ell$ if and only if for all $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \geq \alpha\}$ is closed in X.

[4] A function $f : X \to \mathbb{R}$ is strictly quasi-concave if and only if for all $x, y \in X$ with $f(x) > f(y)$ and for all $\lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) > f(y)$.

[5] A function $f : X \to \mathbb{R}$ is quasi-concave if and only if for all $x, y \in X$ and for all $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$.

Before introducing our new assumption on satiation of preferences, we first define some additional notations.

For each $i \in I$, let $\operatorname{int}_{R_i} A_i$ denote the interior of A_i in the relative topology on $R_i \subset \mathbb{R}^\ell$, that is, for $x_i \in R_i$, we have $x_i \in \operatorname{int}_{R_i} A_i$ if and only if there exists an open ball $B(x_i, r)$ centered at x_i with radius r such that $B(x_i, r) \cap R_i \subset A_i$. Note that $\operatorname{int}_{R_i} A_i = A_i$ if $R_i = A_i$. Let A_i^c and $(\operatorname{int}_{R_i} A_i)^c$ denote the complements of A_i and $\operatorname{int}_{R_i} A_i$ in X_i, that is, $A_i^c = X_i \setminus A_i$ and $(\operatorname{int}_{R_i} A_i)^c = X_i \setminus \operatorname{int}_{R_i} A_i$.

Roughly speaking, if $x_i \in \operatorname{int}_{R_i} A_i$, when consumer $i \in I$ slightly changes his or her consumption plan from x_i so that the resulting consumption bundle x_i' is within R_i, the bundle x_i' will also lie on A_i. In contrast, if $x_i \in A_i \setminus \operatorname{int}_{R_i} A_i$, the resulting consumption bundle $x_i' \in R_i$ may not lie on A_i no matter how small the change is.

We now introduce the following assumption.

Assumption 3. For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap (\operatorname{int}_{R_i} A_i)^c \neq \varnothing$.

Since $(\operatorname{int}_{R_i} A_i)^c = A_i^c \cup (A_i \setminus \operatorname{int}_{R_i} A_i)$, this assumption allows consumer's satiation area to be a subset of the individually rational feasible consumption set, provided that it touches the complement of $\operatorname{int}_{R_i} A_i$ in A_i. In other words, under Assumption 3, we must have $S_i \cap (A_i \setminus \operatorname{int}_{R_i} A_i) \neq \varnothing$ if $S_i \subset A_i$. Note that under Assumptions 1 and 2, the set $A_i \setminus \operatorname{int}_{R_i} A_i$ coincides with the boundary of A_i in the relative topology on R_i.

Assumption 3 generalizes the following two assumptions.

[Nonsatiation] For each $i \in I$, we have $S_i \cap A_i = \varnothing$.

[Weak nonsatiation] For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap A_i^c \neq \varnothing$.

[Nonsatiation] is a standard assumption on preference satiation that ensures the existence of a competitive equilibrium. It excludes the case in which satiation occurs inside the individually rational feasible consumption sets.

[Weak nonsatiation], introduced by Allouch and Le Van (2008a, 2008b), is a generalization of [Nonsatiation]. This assumption allows consumer's satiation points to be inside the individually rational feasible consumption set, provided that at least one satiation point lies outside A_i. However, it excludes the case in which S_i is a subset of A_i, while Assumption 3 does not. Note that [Weak nonsatiation] coincides with [Nonsatiation] when S_i is a singleton for all $i \in I$ with $S_i \neq \varnothing$.

In the following example, only Assumption 3 holds among the above three nonsatiation assumptions.

Example 1. Consider an exchange economy $\mathcal{E}$ with two commodities and two consumers. Let $X_1 = X_2 = \mathbb{R}^2_+$ and $\omega_1 = \omega_2 = (4, 4)$. Consumers' utility functions are as follows.

$$u_1(x_1) = -\|(x_{11}, x_{12}) - (4, 6)\|^2 \quad \text{and} \quad u_2(x_2) = x_{21}.$$

Note that u_1 has a unique satiation point $s_1 = (4, 6)$, while u_2 is never satiated on X_2. [6]

Let $y_2 = (4, 2)$. Then, it is easy to check that the allocations (s_1, y_2) is feasible. Moreover, since

$$u_2(y_2) = 4 = u_2(\omega_2),$$

we have $s_1 \in A_1$. Therefore, neither [Nonsatiation] nor [Weak nonsatiation] holds.

We prove that Assumption 3 holds. Let $\varepsilon_1 = (1, 0)$, and for each $t \in (0, 1]$, let

$$\begin{aligned} z_1(t) &= s_1 + t\varepsilon_1 = (4 + t, 6) \in X_1 \\ z_2(t) &= y_2 - t\varepsilon_1 = (4 - t, 2) \in X_2. \end{aligned}$$

We claim that $z_1(t) \in R_1 \setminus A_1$ for all $t \in (0, 1]$. To see this, note first that we have $z_1(t) + z_2(t) = \sum_{i \in I} \omega_i$ for all $t \in (0, 1]$. Next, since $u_1(z_1(t)) = -t^2 > -4 = u_1(\omega_1)$, we have $z_1(t) \in R_1$ for all $t \in (0, 1]$. Moreover, for all $t \in (0, 1]$, since

$$u_2(z_2(t)) = 4 - t < 4 = u_2(\omega_2),$$

we have $z_2(t) \notin R_2$. Therefore, $z_1(t) \notin A_1$ for all $t \in (0, 1]$.

Since $z_1(t) \in R_1 \setminus A_1$ for all $t \in (0, 1]$ and $z_1(t) \to s_1$ as $t \to 0$, we obtain $s_1 \in A_1 \setminus \operatorname{int}_{R_1} A_1$. Therefore, Assumption 3 holds.

It is easy to check that the allocation $\overline{x} = (s_1, y_2)$ together with the price $\overline{p} = (1, 0)$ is a competitive equilibrium of $\mathcal{E}$.

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[6] Neither the unboundedness of X_i nor the existence of insatiable consumer is essential for the arguments in this example. The same applies to the other examples provided in this paper.

Note that this example also shows that unlike [Weak nonsatiation], Assumption 3 does not coincide with [Nonsatiation] even if S_i is a singleton for all $i \in I$ with $S_i \neq \varnothing$. Our assumption coincides, however, with [Weak nonsatiation] if $R_i = A_i$ for all $i \in I$.

3. Main Results

3.1. Existence of quasi-equilibrium

The purpose of this subsection is to demonstrate the existence of a quasi-equilibrium of $\mathcal{E}$ under Assumptions 1 – 3 (Theorem 2). In its proof, we use the following existence theorem by Allouch and Le Van (2008b). [7]

Theorem 1. *(Allouch and Le Van, 2008b)*
Under Assumptions 1, 2 and [Weak nonsatiation], there exists a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell$ *of* $\mathcal{E}$*.*

The strategy of the proof of our existence theorem is as follows.
First, under our assumptions, we can choose $s = (s_i)_{i \in I} \in S$ so that

$$s_i \in (\operatorname{int}_{R_i} A_i)^c \quad \text{for all} \quad i \in I. \tag{1}$$

Next, for this s, we construct a sequence $\{\omega^\nu\}_{\nu \in \mathbb{N}} = \{(\omega_i^\nu)_{i \in I}\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^{\ell n}$ that satisfies the following properties:

(a) $\omega^\nu \to \omega$ as $\nu \to \infty$,

(b) there exists $\overline{\nu} \in \mathbb{N}$ such that for all $\nu \geq \overline{\nu}$,

(b-1) $\omega_i^\nu \in X_i$ for all $i \in I$, and

(b-2) $s_i \notin R_i^\nu \cap (-\sum_{j \neq i} R_j^\nu + \sum_{k \in I} \{\omega_k^\nu\})$ for all $i \in I$, where

$$R_i^\nu = \{x_i \in X_i : u_i(x_i) \geq u_i(\omega_i^\nu)\}.$$ [8]

[7] In the original version of Theorem 1 (Allouch and Le Van, 2008b, p.5, Theorem 2), instead of Assumption 1 (b), the boundedness of A is assumed.

[8] To be precise, we cannot always find a sequence $\{\omega^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^{\ell n}$ that satisfies all the properties stated above. However, as will be shown later, we may assume without loss of generality that there exists a sequence that satisfies the properties (a) and (b) as far as the existence of quasi-equilibrium matters.

We then define an auxiliary economy $\mathcal{E}^\nu$ by $\mathcal{E}^\nu = (\mathbb{R}^\ell, (X_i, u_i, \omega_i^\nu)_{i\in I})$ for each $\nu \geq \overline{\nu}$ (the economy $\mathcal{E}^\nu$ differs from the initial economy only in its initial endowments). By the definition, each $\mathcal{E}^\nu$ satisfies all the assumptions in Theorem 1. Especially, [Weak nonsatiation] holds by the property (b-2) of $\{\omega^\nu\}_{\nu\in\mathbb{N}}$.

Therefore, we obtain a sequence $\{(\overline{x}^\nu, \overline{p}^\nu)\}_{\nu\geq\overline{\nu}} \subset X \times \mathbb{R}^\ell$ each term of which is a quasi-equilibrium of $\mathcal{E}^\nu$. Under our assumptions, we may assume that the sequence has a limit point, and we can prove that the point is a quasi-equilibrium of the original economy.

Next lemma shows that for a fixed $s \in S$ that satisfies (1), we can find a sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ with certain properties. It is used in our main existence theorem to construct the sequence $\{\omega^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^{\ell n}$ described above.

Lemma 1. *Suppose that Assumptions 1 and 2 hold, and suppose that there exists $s = (s_i)_{i\in I} \in S$ that satisfies*

$$s_i \in (\operatorname{int}_{R_i} A_i)^c \quad \text{for all} \quad i \in I.$$

Then, there exist $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ and $\overline{\nu} \in \mathbb{N}$ such that $\nu\varepsilon^\nu \to 0$ as $\nu \to \infty$, and for every $\nu \geq \overline{\nu}$ and $i \in I$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}).$$

Proof. Let $s = (s_i)_{i\in I} \in S$ be the element that satisfies $s_i \in (\operatorname{int}_{R_i} A_i)^c$ for all $i \in I$. Put $I_{bd} = \{i \in I : s_i \in A_i \setminus \operatorname{int}_{R_i} A_i\}$ and $I_{out} = I \setminus I_{bd} = \{i \in I : s_i \notin A_i\}$.

In the following, we divide the proof into several cases, in each of which we construct a sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ that satisfies the properties stated in the lemma.

Case 1. $I_{bd} = \varnothing$.

First, if $I_{bd} = \varnothing$ (equivalently, $I_{out} = I$), it is clear that the sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ defined by $\varepsilon^\nu = 0$ for all $\nu \in \mathbb{N}$ satisfies the desired properties.

Case 2. $I_{bd} \neq \varnothing$.

We first consider the sequence $\{\varepsilon_{out}^{\nu}\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^{\ell}$ defined by

$$\varepsilon_{out}^{\nu} = \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h - \omega_h) \quad \text{for each} \quad \nu \in \mathbb{N}.$$

It is clear that $\varepsilon_{out}^{\nu} \to 0$ and $\nu\varepsilon_{out}^{\nu} \to 0$ as $\nu \to \infty$.

Then, for $i \in I_{out}$, we have the following claim.

Claim 1. *For each $i \in I_{out}$, there exists $\overline{\nu}_i \in \mathbb{N}$ such that*

$$s_i \notin R_i \cap (G_i - \{\varepsilon_{out}^{\nu}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}_i.$$

Proof of Claim 1. First, for each $i \in I_{out}$, since $s_i \notin A_i$ and $s_i \in R_i$, we must have $s_i \notin G_i$. Then, since G_i is closed in $\mathbb{R}^{\ell}$, there exists a positive real number $r_i > 0$ such that $B(s_i, r_i) \cap G_i = \varnothing$. Since $\varepsilon_{out}^{\nu} \to 0$ as $\nu \to \infty$, there exists $\overline{\nu}_i$ such that $s_i + \varepsilon_{out}^{\nu} \in B(s_i, r_i)$ for all $\nu \geq \overline{\nu}_i$, which implies that $s_i \notin G_i - \{\varepsilon_{out}^{\nu}\}$ for all $\nu \geq \overline{\nu}_i$. $\square$

With respect to $i \in I_{bd}$, we have the following claim.

Claim 2. *For each $i \in I_{bd}$, either (a) there exists $\overline{\nu}_i \in \mathbb{N}$ such that*

$$s_i \notin R_i \cap (G_i - \{\varepsilon_{out}^{\nu}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}_i$$

or (b) there exists $\overline{\nu}_i' \in \mathbb{N}$ such that

$$s_i \in R_i \cap (G_i - \{\varepsilon_{out}^{\nu}\}) \quad \textit{for all} \quad \nu \geq \overline{\nu}_i'.$$

The proof of Claim 2 is given in the Appendix.

Next, let I_{in} be the subset of I_{bd} such that $i \in I_{in}$ if and only if there exists $\overline{\nu}_i \in \mathbb{N}$ that satisfies

$$s_i \in R_i \cap (G_i - \{\varepsilon_{out}^{\nu}\}) \quad \text{for all} \quad \nu \geq \overline{\nu}_i.$$

By the definition of I_{in} and Claims 1 and 2, there exists $\overline{\nu}_{out}$ such that for all $i \in I \setminus I_{in}$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon_{out}^{\nu}\}) \quad \text{for all} \quad \nu \geq \overline{\nu}_{out}.$$

Then, we have two cases.

Case 2-A. $I_{in} = \varnothing$.

It is clear that the sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ defined by $\varepsilon^\nu = \varepsilon^\nu_{out}$ for each $\nu \in \mathbb{N}$ satisfies all the properties in the statement of the lemma.

Case 2-B. $I_{in} \neq \varnothing$.

For simplicity of notation, we assume without loss of generality that $I_{in} = \{1, 2, \cdots, M\}$, where $M = |I_{in}| \leq n$.

In view of the definition of I_{in}, there exists $\overline{\nu}_{in}$ such that for all $i \in I_{in}$,

$$s_i \in R_i \cap (G_i - \{\varepsilon^\nu_{out}\}) \quad \text{for all} \quad \nu \geq \overline{\nu}_{in}.$$

Put $\overline{\nu} = \max\{\overline{\nu}_{out}, \overline{\nu}_{in}\}$.

We construct the sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}}$ in several steps.

First, for each fixed $\nu \geq \overline{\nu}$, we inductively construct M vectors $\varepsilon^\nu_1, \cdots, \varepsilon^\nu_M \in \mathbb{R}^\ell$ that satisfy the following properties:

For each $m \in I_{in}$,

(i) $s_m + \varepsilon^\nu_m \in R_m \setminus A_m$ and $\|\varepsilon^\nu_m\| < 1/2^m \nu^{2m+2}$, and

(ii) for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m} 2^q \nu^{2q} \{\varepsilon^\nu_q\} \right).$$

We first construct ε^ν_1 as follows.

Since $\nu \geq \overline{\nu}$ $(\geq \overline{\nu}_{out})$, there exists a positive real number r^ν_1 such that

$$B(s_i, r^\nu_1) \cap (G_i - \{\varepsilon^\nu_{out}\}) = \varnothing \quad \text{for all} \quad i \in I \setminus I_{in}.$$

Then, since $s_1 \in A_1 \setminus \operatorname{int}_{R_1} A_1$, there exists $\varepsilon^\nu_1 \in \mathbb{R}^\ell \setminus \{0\}$ such that

$$s_1 + \varepsilon^\nu_1 \in R_1 \setminus A_1 \quad \text{and} \quad \|\varepsilon^\nu_1\| < \min\left\{ \frac{r^\nu_1}{2\nu^2}, \frac{1}{2\nu^4} \right\}.[9]$$

[9]Recall that for $x_i \in A_i$, we have $x_i \in A_i \setminus \operatorname{int}_{R_i} A_i$ if and only if

$$B(x_i, r) \cap R_i \not\subset A_i$$

for any positive real number r.

Since $2\nu^2\|\varepsilon_1^\nu\| < r_1^\nu$, we have

$$s_i \notin R_i \cap (G_i - \{\varepsilon_{out}^\nu\} - 2\nu^2\{\varepsilon_1^\nu\}) \quad \text{for all} \quad i \in I \setminus I_{in}. \tag{2}$$

We need to show that (2) also holds for $i = m = 1$.

Claim 3.

$$s_1 \notin R_1 \cap (G_1 - \{\varepsilon_{out}^\nu\} - 2\nu^2\{\varepsilon_1^\nu\}).$$

The proof of Claim 3 is given in the Appendix.

Let $m \in I_{in}$ with $m \geq 2$, and suppose that $\varepsilon_1^\nu, \cdots, \varepsilon_{m-1}^\nu$ are the vectors that satisfy properties (i) and (ii) for each m. We construct ε_m^ν as follows.

First, by property (ii) with respect to $m-1$, we have, for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon_{out}^\nu\} - \sum_{q=1}^{m-1} 2^q \nu^{2q} \{\varepsilon_q^\nu\}\right), \tag{3}$$

Note that for all $q \in \{1, \cdots, m-1\}$, by the first part of property (i) and the convexity of R_q, we have $s_q + \lambda\varepsilon_q^\nu \in R_q$ for all $\lambda \in [0,1]$.

By (3), there exists a positive real number r_m^ν such that for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$B(s_i, r_m^\nu) \cap \left(G_i - \{\varepsilon_{out}^\nu\} - \sum_{q=1}^{m-1} 2^q \nu^{2q} \{\varepsilon_q^\nu\}\right) = \varnothing.$$

Since $s_m \in A_m \setminus \mathrm{int}_{R_m} A_m$, we can choose $\varepsilon_m^\nu \in \mathbb{R}^\ell$ so that

$$s_m + \varepsilon_m^\nu \in R_m \setminus A_m \quad \text{and} \quad \|\varepsilon_m^\nu\| < \min\left\{\frac{r_m^\nu}{2^m \nu^{2m}}, \frac{1}{2^m \nu^{2m+2}}\right\}.$$

Since $2^m \nu^{2m}\|\varepsilon_m^\nu\| < r_m^\nu$, we have, for all $i \in (I \setminus I_{in}) \cup \{1, \cdots, m-1\}$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon_{out}^\nu\} - \sum_{q=1}^{m} 2^q \nu^{2q} \{\varepsilon_q^\nu\}\right), \tag{4}$$

We claim that (4) also holds for $i = m$.

Claim 4.

$$s_m \notin R_m \cap \left(G_m - \{\varepsilon^\nu_{out}\} - \sum_{q=1}^{m} 2^q \nu^{2q} \{\varepsilon^\nu_q\} \right).$$

The proof of Claim 4 is given in the Appendix.

Thus, we conclude that for each $\nu \geq \overline{\nu}$, there exist M vectors $\varepsilon^\nu_1, \cdots, \varepsilon^\nu_M \in \mathbb{R}^\ell$ that satisfy the properties (i) and (ii). Note that by property (ii) with respect to $m = M$,

$$s_i \notin R_i \cap \left(G_i - \{\varepsilon^\nu_{out}\} - \sum_{m \in I_{in}} 2^m \nu^{2m} \{\varepsilon^\nu_m\} \right) \quad \text{for all} \quad i \in I. \tag{5}$$

We now define a sequence $\{\varepsilon^\nu_{in}\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^\ell$ by

$$\varepsilon^\nu_{in} = \begin{cases} 1 & \text{for} \quad \nu < \overline{\nu} \\ \sum_{m \in I_{in}} 2^m \nu^{2m} \varepsilon^\nu_m & \text{for} \quad \nu \geq \overline{\nu}. \end{cases}$$

Since

$$\|\varepsilon^\nu_{in}\| \leq \sum_{m \in I_{in}} 2^m \nu^{2m} \|\varepsilon^\nu\| < \frac{M}{\nu^2} \quad \text{for all} \quad \nu \geq \overline{\nu},$$

we have $\nu\varepsilon^\nu_{in} \to 0$ as $\nu \to \infty$ (recall that for all $\nu \geq \overline{\nu}$ and $m \in I_{in}$, by the second part of property (i), we have $2^m \nu^{2m} \|\varepsilon^\nu_m\| < 1/\nu^2$).

Finally, define a sequence $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{R}^\ell$ by

$$\varepsilon^\nu = \varepsilon^\nu_{out} + \varepsilon^\nu_{in}.$$

Then, from the definition, we have $\nu\varepsilon^\nu \to 0$ as $\nu \to \infty$. Moreover, by (5), for all $\nu \geq \overline{\nu}$, we have

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}) \quad \text{for all} \quad i \in I,$$

which completes the proof. □

Let $\{\varepsilon^\nu\}_{\nu \in \mathbb{N}}$ be a sequence that satisfies the properties stated in Lemma 1. Next Lemma shows that we may assume that $\sum_{i \in I} \omega_i - \nu\varepsilon^\nu \in \sum_{i \in I} R_i$ for sufficiently large $\nu \in \mathbb{N}$ as far as the existence of quasi-equilibrium matters.

Lemma 2. *Suppose Assumptions 1 and 2 hold. Suppose that there exists a sequence* $\{\delta^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ *such that* $\delta^\nu \to 0$ *as* $\nu \to \infty$*, and*

$$\sum_{i\in I} \omega_i - \delta^\nu \notin \sum_{i\in I} R_i \quad \textit{for all} \quad \nu \in \mathbb{N}.$$

Then, there exists a ℓ*-dimensional vector* $\overline{p} \neq 0$ *such that*

$$\overline{p} \cdot R_i \geq \overline{p} \cdot \omega_i \quad \textit{for all} \quad i \in I.^{10}$$

Therefore, $(\omega, \overline{p}) \in X \times \mathbb{R}^\ell$ *is a quasi-equilibrium of* $\mathcal{E}$*.*

Proof. If $\sum_{i\in I} \omega_i - \delta^\nu \notin \sum_{i\in I} R_i$ for all $\nu \in \mathbb{N}$, we have $\sum_{i\in I} \omega_i \in \mathrm{bd}(\sum_{i\in I} R_i)$. Since $\sum_{i\in I} R_i$ is convex under Assumptions 1 and 2, by the support theorem (Florenzano and Le Van, 2001, p25, Corollary 2.1.1), there exists $\overline{p} \neq 0$ such that

$$\overline{p} \cdot z \geq \overline{p} \cdot \sum_{i\in I} \omega_i \quad \text{for all} \quad z \in \sum_{i\in I} R_i.$$

Take arbitrary $i \in I$ and $x_i \in R_i$. Since $x_i + \sum_{j\neq i} \omega_j \in \sum_{k\in I} R_k$, we have

$$\overline{p} \cdot x_i + \overline{p} \cdot \sum_{j\neq i} \omega_j \geq \overline{p} \cdot \omega_i + \overline{p} \cdot \sum_{j\neq i} \omega_j,$$

and thus,

$$\overline{p} \cdot x_i \geq \overline{p} \cdot \omega_i.$$

Therefore, we have

$$\overline{p} \cdot R_i \geq \overline{p} \cdot \omega_i \quad \text{for all} \quad i \in I,$$

which completes the proof. □

We now state and prove our main existence theorem.

Theorem 2. *Under Assumptions 1 – 3, there exists a quasi-equilibrium* $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$*.*

[10]By "$\overline{p} \cdot R_i \geq \overline{p} \cdot \omega_i$", we mean $\overline{p} \cdot x_i \geq \overline{p} \cdot \omega_i$ for all $x_i \in R_i$.

Proof. By Assumptions 1 (a), 1 (b), 2 (a) and 3, there exists $s = (s_i)_{i\in I} \in S$ such that $s_i \in (\mathrm{int}_{R_i} A_i)^c$ for all $i \in I$.

Then, by Lemma 1, there exist a sequence $\{\varepsilon^\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^\ell$ and a natural number $\overline{\nu}_1 \in \mathbb{N}$ such that $\nu\varepsilon^\nu \to 0$ as $\nu \to \infty$, and for every $\nu \geq \overline{\nu}_1$ and $i \in I$,

$$s_i \notin R_i \cap (G_i - \{\varepsilon^\nu\}). \tag{6}$$

Suppose that the sequence $\{\nu\varepsilon^\nu\}_{\nu\in\mathbb{N}}$ contains a subsequence $\{\nu_\mu\varepsilon^{\nu_\mu}\}_{\mu\in\mathbb{N}}$ that satisfies

$$\sum_{i\in I} \omega_i - \nu_\mu\varepsilon^{\nu_\mu} \notin \sum_{i\in I} R_i \quad \text{for all} \quad \mu \in \mathbb{N}.$$

Then, by Lemma 2, $(\omega, \overline{p}) \in X \times \mathbb{R}^\ell$ is a quasi-equilibrium of $\mathcal{E}$.

Therefore, we may suppose without loss of generality that there exists $\overline{\nu}_2 \in \mathbb{N}$ such that for all $\nu \geq \overline{\nu}_2$,

$$\sum_{i\in I} \omega_i - \nu\varepsilon^\nu \in \sum_{i\in I} R_i.$$

By this relation, for each $\nu \geq \overline{\nu} = \max\{\overline{\nu}_1, \overline{\nu}_2\}$, there exists $x^\nu = (x_i^\nu)_{i\in I} \in \prod_{i\in I} R_i$ such that $\sum_{i\in I} x_i^\nu = \sum_{i\in I} \omega_i - \nu\varepsilon^\nu$. Note that since R_i is compact, the sequence $\{x_i^\nu\}_{\nu\geq\overline{\nu}} \subset R_i$ is bounded for each $i \in I$.

For each $\nu \geq \overline{\nu}$ and $i \in I$, let

$$\omega_i^\nu = \left(1 - \frac{1}{\nu}\right)\omega_i + \frac{1}{\nu}x_i^\nu.$$

Then, we have $\omega_i^\nu \in R_i$ by the convexity of R_i, and

$$\sum_{i\in I} \omega_i^\nu = \left(1 - \frac{1}{\nu}\right)\sum_{i\in I} \omega_i + \frac{1}{\nu}\sum_{i\in I} x_i^\nu = \sum_{i\in I} \omega_i - \varepsilon^\nu. \tag{7}$$

Moreover, $\omega_i^\nu \to \omega_i$ as $\nu \to \infty$ for each $i \in I$. Indeed, since $\{x_i^\nu\}_{\nu\in\mathbb{N}}$ is bounded,

$$\|\omega_i - \omega_i^\nu\| \leq \frac{1}{\nu}\|\omega_i\| + \frac{1}{\nu}\|x_i^\nu\| \to 0 \quad \text{as} \quad \nu \to \infty.$$

We now define, for each $\nu \geq \overline{\nu}$, an auxiliary economy $\mathcal{E}^\nu$ by

$$\mathcal{E}^\nu = (\mathbb{R}^\ell, (X_i, u_i, \omega_i^\nu)_{i\in I}).$$

Note that $\mathcal{E}^\nu$ differs from $\mathcal{E}$ only in its initial endowments.

Then, by the definition, each $\mathcal{E}^\nu$ satisfies all the assumptions in Theorem 1. Especially, each $\mathcal{E}^\nu$ satisfies [Weak nonsatiation]. To see this, note first that for each $i \in I$, by (6) and (7),

$$\begin{aligned} s_i \notin\ & G_i - \{\varepsilon^\nu\} \\ &= -\sum_{j \neq i} R_j + \sum_{k \in I} \{\omega_k\} - \{\varepsilon^\nu\} \\ &= -\sum_{j \neq i} R_j + \sum_{k \in I} \{\omega_k^\nu\}. \end{aligned} \tag{8}$$

Since $u_i(\omega_i^\nu) \geq u_i(\omega_i)$ by the quasi-concavity of u_i, we have $R_i^\nu \subset R_i$ for each $i \in I$, where

$$R_i^\nu = \{x_i \in X_i : u_i(x_i) \geq u_i(\omega_i^\nu)\}.$$

Therefore,

$$-\sum_{j \neq i} R_j^\nu + \sum_{k \in I} \{\omega_k^\nu\} \subset -\sum_{j \neq i} R_j + \sum_{k \in I} \{\omega_k^\nu\}.$$

Finally, by this relation and (8),

$$s_i \notin R_i^\nu \cap \left(-\sum_{j \neq i} R_j^\nu + \sum_{k \in I} \{\omega_k^\nu\} \right) = A_i^\nu,$$

where A_i^ν denotes the individually rational feasible consumption set of consumer $i \in I$ in $\mathcal{E}^\nu$.

Therefore, by Theorem 1, each $\mathcal{E}^\nu$ ($\nu \geq \overline{\nu}$) admits a quasi-equilibrium $(\overline{x}^\nu, \overline{p}^\nu) \in X \times \mathbb{R}^\ell \setminus \{0\}$. In view of Definition 2, we may assume without loss of generality that $\overline{p}^\nu \in S(0,1) = \{p \in \mathbb{R}^\ell : \|p\| = 1\}$ for all $\nu \geq \overline{\nu}$.

We now obtain a sequence $\{(\overline{x}^\nu, \overline{p}^\nu)\}_{\nu \geq \overline{\nu}} \subset X \times S(0,1)$ each term of which is a quasi-equilibrium of $\mathcal{E}^\nu$. Since $X \times S(0,1)$ is compact, we may assume without loss of generality that the sequence has a limit point $(\overline{x}, \overline{p}) \in X \times S(0,1)$. We prove that $(\overline{x}, \overline{p})$ is a quasi-equilibrium of the original economy.

We fist show that (a-2) of Definition 2 holds. Suppose that for some $i \in I$, there exists $x_i \in X_i$ with

$$u_i(x_i) > u_i(\overline{x}_i) \quad \text{and} \quad \overline{p} \cdot x_i < \overline{p} \cdot \omega_i.$$

Then, since $(\overline{x}^\nu, \overline{p}^\nu) \to (\overline{x}, \overline{p})$ and $\omega_i^\nu \to \omega_i$ as $\nu \to \infty$, and u_i is upper semicontinuous, we have

$$u_i(x_i) > u_i(\overline{x}_i^\nu) \quad \text{and} \quad \overline{p}^\nu \cdot x_i < \overline{p}^\nu \cdot \omega_i^\nu$$

for sufficiently large ν. However, this contradicts with the fact that $(\overline{x}^\nu, \overline{p}^\nu)$ is a quasi-equilibrium of $\mathcal{E}^\nu$. Thus, (a-2) of Definition 2 holds. It is easy to check that $(\overline{x}, \overline{p})$ satisfies (a-1) and (b) of Definition 2.

Therefore, we conclude that $(\overline{x}, \overline{p}) \in X \times S(0,1)$ is a quasi-equilibrium of the original economy $\mathcal{E}$. $\square$

3.2. Existence of competitive equilibrium

There are several known sets of assumptions under which a quasi equilibrium is a competitive equilibrium (see, for example, Geistdoerfer-Florenzano, 1982). In this paper, we employ the simplest one:

Assumption 4. For all $i \in I$,

(a) $\omega_i \in \operatorname{int} X_i$, and

(b) u_i is continuous on X_i.

We now establish the existence of a competitive equilibrium under Assumption 3.

Theorem 3. *Under Assumptions 1 – 4, there exists a competitive equilibrium* $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$.

Proof. By Theorem 2, there exists a quasi-equilibrium $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ of $\mathcal{E}$. We prove that $(\overline{x}, \overline{p})$ is a competitive equilibrium of $\mathcal{E}$.

It is clear that $(\overline{x}, \overline{p})$ satisfies (a-1) and (b) of Definition 1. Suppose that (a-2) of Definition 1 does not hold for some $i \in I$. Then, there exists $x_i \in X_i$ such that

$$u_i(x_i) > u_i(\overline{x}_i) \quad \text{and} \quad \overline{p} \cdot x_i = \overline{p} \cdot \omega_i.$$

Since $\overline{p} \neq 0$ and $\omega_i \in \operatorname{int} X_i$, there exists $y_i \in X_i$ such that $\overline{p} \cdot y_i < \overline{p} \cdot \omega_i$. Let $x_i(t) = tx_i + (1-t)y_i$ for $t \in (0,1)$. It is clear that for all $t \in (0,1)$,

$$x_i(t) \in X_i \quad \text{and} \quad \overline{p} \cdot x_i(t) < \overline{p} \cdot \omega_i.$$

Moreover, since $u_i(x_i) > u_i(\overline{x}_i)$ and u_i is continuous on X_i, we have $u_i(x_i(t)) > u_i(\overline{x}_i)$ for t sufficiently close to 1. However, this contradicts with the fact that $(\overline{x}, \overline{p})$ is a quasi-equilibrium of $\mathcal{E}$.

Therefore, we conclude that $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ is a competitive equilibrium of $\mathcal{E}$. $\square$

4. Alternative assumption

In this section, we introduce an alternative to Assumption 3 and provide some related results.

Consider the following assumption.

Assumption 5. For each $i \in I$, if $S_i \neq \varnothing$, we have $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$.

The symbol $\operatorname{int}_{X_i} F_i$ denotes the interior of F_i in the relative topology on X_i. [11] Assumption 5 allows S_i to be a subset of the individually feasible consumption set F_i, provided that it touches the complement of F_i in X_i. This assumption is a generalization of Sato's (2008) nonsatiation assumption, which asserts that $S_i \cap \operatorname{int}_{X_i} F_i = \varnothing$ for each $i \in I$.

By replacing A_i by F_i and R_i by X_i for all $i \in I$ in the statements and proofs of all the propositions provided in 3.1 (including Theorem 1), we obtain the existence of a quasi-equilibrium under Assumption 5.

Corollary 1. *Under Assumptions 1, 2 and 5, there exists a quasi-equilibrium* $(\overline{x}, \overline{p}) \in X \times \mathbb{R}^\ell \setminus \{0\}$ *of* $\mathcal{E}$. [12]

Assumptions 3 and 5 do not imply each other in general. Indeed, Assumption 5 does not hold in Example 1 in 2.2 (where s_1 lies on $\operatorname{int}_{X_1} F_1$). In contrast, in Example 2 below, we will observe that only Assumption 5 holds.

Example 2. Consider an exchange economy $\mathcal{E}$ with two commodities and two consumers. Let $X_1 = \{x_1 \in \mathbb{R}^2 : 0 \leq x_{11} \leq 10 \text{ and } x_{12} \geq 0\}$ and $X_2 = \mathbb{R}^2_+$. Let $\omega_1 = (0, 10)$ and $\omega_2 = (10, 0)$. Consumers' utility functions are as follows.

$$u_1(x_1) = \begin{cases} -|10 - x_{12}| & \text{if} \quad x_{12} \neq 10 \\ -|5 - x_{11}| + 5 & \text{if} \quad x_{12} = 10 \end{cases}$$

$$u_2(x_2) = x_{22}.$$

[11] Roughly speaking, if $x_i \in \operatorname{int}_{X_i} F_i$, every consumption bundle nearby it also lies on F_i.

[12] In this corollary, Assumption 1 (b) can be weakened to the boundedness of F by the standard truncation technique.

Note that u_1 is satiated at $s_1 = (5, 10)$ and u_2 is never satiated on X_2.

By the above definitions, it is easy to check that

$$A_1 = \{x_1 \in X_1 : 0 \leq x_{11} \leq 10 \text{ and } x_{12} = 10\} = R_1$$

and $s_1 \in A_1$. Since $A_i \setminus \operatorname{int}_{R_i} A_i = \varnothing$ when $A_i = R_i$, Assumption 3 does not hold. However, since $s_1 \in F_1 \setminus \operatorname{int}_{X_1} F_1$ (note that s_1 requires the total amount of the second good in the economy while consumer $i \in I$ can consume more of it), Assumption 5 holds.

However, Assumption 5 implies Assumption 3 if consumers' utility functions are continuous and not satiated at the initial endowments.

Proposition 1. *Suppose that u_i is continuous on X_i and $\omega_i \notin S_i$ for all $i \in I$, then Assumption 5 implies Assumption 3.* [13]

Proof. It suffices to show that for all $i \in I$, if $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$, then, $S_i \cap (\operatorname{int}_{R_i} A_i)^c \neq \varnothing$. Suppose that $S_i \cap (\operatorname{int}_{X_i} F_i)^c \neq \varnothing$ and $S_i \cap (\operatorname{int}_{R_i} A_i)^c = \varnothing$ for some $i \in I$. Let $s_i \in S_i \cap (\operatorname{int}_{X_i} F_i)^c$. Then, by the supposition, we have $s_i \in \operatorname{int}_{R_i} A_i$.

Since $s_i \in \operatorname{int}_{R_i} A_i$, there exists a positive real number r_1 such that

$$B(s_i, r_1) \cap R_i \subset A_i.$$

Moreover, since u_i is continuous on X_i and $u_i(s_i) > u_i(\omega_i)$, there exists a positive real number r_2 such that

$$B(s_i, r_2) \cap X_i \subset R_i.$$

Let $r = \min\{r_1, r_2\}$. Then, from the above two relations,

$$B(s_i, r) \cap X_i \subset B(s_i, r_1) \cap R_i \subset A_i \subset F_i.$$

Therefore, $s_i \in \operatorname{int}_{X_i} F_i$, which is a contradiction. □

[13] Example 1 in 2.2 shows that the converse of the statement is not true.

5. Concluding Remark

As a concluding remark, we compare Assumption 3 with the assumption introduced by Won and Yannelis (2006).

Won and Yannelis (2006) establish the existence of competitive equilibrium with satiation in more general settings than ours. For example, in their analysis, individually rational feasible consumption sets do not need to be bounded and consumers' preferences are allowed to be non-ordered. Moreover, they introduce an assumption that allows each consumer's satiation area S_i to be a subset of $\mathrm{int}_{R_i} A_i$, while our assumption does not apply to such a case. Therefore, their assumption does not imply Assumption 3. Nevertheless, as shown below, our assumption does not imply Won and Yannelis's assumption either.

To simplify the arguments, we consider the case in which if $S_i \neq \varnothing$ for some $i \in I$, it consists of a unique element $s_i \in X_i$.

For an allocation $x \in X$, let $I_s(x) = \{i \in I : x_i \in S_i\}$ and $I_{ns}(x) = I \setminus I_s(x)$. For a consumption bundle $x_i \in X_i$, let $P_i(x_i) = \{y_i \in X_i : u_i(y_i) > u_i(x_i)\}$. Then, Won and Yannelis's (2006) condition reduces to the following form:

> Let $x = (x_i)_{i \in I} \in A$ with $I_s(x) \neq \varnothing$ and $I_{ns}(x) \neq \varnothing$. Then, for each $p \in \mathbb{R}^\ell \setminus \{0\}$ that satisfies $p \cdot P_j(x_j) > p \cdot x_j$ for all $j \in I_{ns}(x)$, we have $p \cdot s_i \geq p \cdot \omega_i$ for all $i \in I_s(x)$. [14]

We now consider the following example.

Example 3. Consider an exchange economy $\mathcal{E}$ with two commodities and three consumers. Let $X_i = \mathbb{R}^2_+$ for all $i \in I = \{1, 2, 3\}$. Let $\omega_1 = \omega_2 = (2, 2)$ and $\omega_3 = (4, 4)$. Consumers' utility functions are as follows.

$$u_1(x_1) = -\|(x_{11}, x_{12}) - (8, 0)\|^2$$

$$u_2(x_2) = -\|(x_{21}, x_{22}) - (0, 8)\|^2$$

$$u_3(x_3) = x_{32} - x_{31}$$

Note that $s_1 = (8, 0)$ $(\neq \omega_1)$ and $s_2 = (0, 8)$ $(\neq \omega_2)$ are the unique satiation points of consumers 1 and 2. However, u_3 is never satiated on X_3. Note also that u_i is continuous on X_i for each $i \in I$.

[14] Won and Yannelis, 2006, p.4, Assumption **S5**.

Consider an allocation $x = (s_1, s_2, y_3) \in X$, where $y_3 = (0, 0) \in X_3$. It is clear that x is individually rational feasible and $I_s(x) = \{1, 2\}$ and $I_{ns} = \{3\}$.

By the definition of u_3, we have $P_3(y_3) = \{x_3 \in X_3 : x_{32} > x_{31}\}$. Therefore, for a price $p = (-1, 1)$, we have

$$p \cdot P_3(y_3) > 0 = p \cdot y_3.$$

However, since $p \cdot s_1 = -8 < 0 = p \cdot \omega_1$ for consumer 1, Won and Yannelis's assumption does not hold.

To prove that Assumption 3 holds, we first observe that Assumption 5 holds. Indeed, since s_1 requires the total amount of the first good in the economy while consumer 1 can consume more of it, we have $s_1 \in F_1 \setminus \mathrm{int}_{X_1} F_1$. Likewise, we have $s_2 \in F_2 \setminus \mathrm{int}_{X_2} F_2$. Therefore, this economy satisfies Assumption 5. Then, by Proposition 1, we conclude that Assumption 3 holds.

Appendix

Proof of Claim 2. First, if $\sum_{h \in I_{bd}} (s_h - \omega_h) = 0$, case (b) clearly holds. Thus, in the following, we suppose that $\sum_{h \in I_{bd}} (s_h - \omega_h) \neq 0$.

Suppose that the assertion of the lemma is not true. Then, for an arbitrarily chosen $\nu \in \mathbb{N}$, there exist $\nu', \nu'' \geq \nu$ such that

$$s_i \notin R_i \cap \left(G_i - \frac{1}{(\nu')^2} \sum_{h \in I_{bd}} \{s_h - \omega_h\} \right) \tag{9}$$

and

$$s_i \in R_i \cap \left(G_i - \frac{1}{(\nu'')^2} \sum_{h \in I_{bd}} \{s_h - \omega_h\} \right). \tag{10}$$

Without loss of generality, we may assume that $\nu' > \nu''$.

Then, by (10), there exists $(x_j)_{j \neq i} \in \prod_{j \neq i} R_j$ such that

$$s_i = -\sum_{j \neq i} x_j + \sum_{k \in I} \omega_k - \frac{1}{(\nu'')^2} \sum_{h \in I_{bd}} (s_h - \omega_h). \tag{11}$$

Since $s_i \in A_i = R_i \cap G_i$, there exists $(y_j)_{j \neq i} \in \prod_{j \neq i} R_j$ such that

$$s_i = -\sum_{j \neq i} y_j + \sum_{k \in I} \omega_k. \tag{12}$$

Then, $(\nu''/\nu')^2\times$(11)+$(1-(\nu''/\nu')^2)\times$(12) yields,

$$s_i = -\sum_{j\neq i} z_j + \sum_{k\in I}\omega_k - \frac{1}{(\nu')^2}\sum_{h\in I_{bd}}(s_h-\omega_h),$$

where

$$z_j = \left(\frac{\nu''}{\nu'}\right)^2 x_j + \left(1-\left(\frac{\nu''}{\nu'}\right)^2\right) y_j \in R_j \quad \text{for each} \quad j\neq i.$$

Therefore,

$$s_i \in R_i \cap \left(G_i - \frac{1}{(\nu')^2}\sum_{h\in I_{bd}}\{s_h-\omega_h\}\right),$$

which contradicts with (9). □

Proof of Claim 3. Suppose that the assertion of the claim is not true. Then, there exists $(x_j)_{j\neq 1} \in \prod_{j\neq 1} R_j$ such that

$$s_1 = -\sum_{j\neq 1} x_j + \sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h) - 2\nu^2\varepsilon_1^\nu.$$

Since $s_1 \in R_1 \cap (G_1 - \{\varepsilon_{out}^\nu\})$ (recall that $\nu \geq \overline{\nu} \geq \overline{\nu}_{in}$), there exists $(y_j)_{j\neq 1} \in \prod_{j\neq 1} R_j$ such that

$$s_1 = -\sum_{j\neq 1} y_j + \sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h).$$

Therefore, we have

$$s_1 = -\sum_{j\neq 1} z_j + \sum_{k\in I}\omega_k - \frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h) - 2\varepsilon_1^\nu, \tag{13}$$

where

$$z_j = \frac{1}{\nu^2}x_j + \left(1-\frac{1}{\nu^2}\right)y_j \in R_j \quad \text{for all} \quad j\neq 1.$$

Moreover, since $s_1 \in A_1 = R_1 \cap G_1$, there exists $(t_j)_{j\neq 1} \in \prod_{j\neq 1} R_j$ such that

$$s_1 = -\sum_{j\neq 1} t_j + \sum_{k\in I}\omega_k. \tag{14}$$

Multiplying (14) by $(1-(1/\nu)^2)$, we have

$$\left(1-\frac{1}{\nu^2}\right)s_1=-\sum_{j\neq 1}\left(1-\frac{1}{\nu^2}\right)t_j+\left(1-\frac{1}{\nu^2}\right)\sum_{k\in I}\omega_k.$$

By adding this equation (and $0=-(1/\nu^2)\sum_{h\in I_{out}}(\omega_h-\omega_h)$) to (13) and rearranging it, we obtain

$$2(s_1+\varepsilon_1^\nu)=-\sum_{j\neq 1}a_j+2\sum_{k\in I}\omega_k,$$

where for $j\in I_{out}$,

$$a_j=z_j+\left[\frac{1}{\nu^2}\omega_j+\left(1-\frac{1}{\nu^2}\right)t_j\right],$$

and for $j\in I_{bd}\setminus\{1\}$,

$$a_j=z_j+\left[\frac{1}{\nu^2}s_j+\left(1-\frac{1}{\nu^2}\right)t_j\right].$$

Therefore,

$$s_1+\varepsilon_1^\nu=-\sum_{j\neq 1}\frac{1}{2}a_j+\sum_{k\in I}\omega_k.$$

Since $(1/2)a_j\in R_j$ for all $j\neq 1$, we have $s_1+\varepsilon_1^\nu\in A_1$, which contradicts with our choice of ε_1^ν. □

Proof of Claim 4. Suppose that there exists $(x_j)_{j\neq m}\in\prod_{j\neq m}R_j$ such that

$$s_m=-\sum_{j\neq m}x_j+\sum_{k\in I}\omega_k-\frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h)-\sum_{q=1}^{m}2^q\nu^{2q}\varepsilon_q^\nu.$$

Since $s_m\in R_m\cap(G_m-\{\varepsilon_{out}^\nu\})$, there exists $(y_j)_{j\neq m}\in\prod_{j\neq m}R_j$ such that

$$s_m=-\sum_{j\neq m}y_j+\sum_{k\in I}\omega_k-\frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h).$$

Therefore, we have

$$s_m=-\sum_{j\neq m}z_j+\sum_{k\in I}\omega_k-\frac{1}{\nu^2}\sum_{h\in I_{bd}}(s_h-\omega_h)-\sum_{q=1}^{m}2^{q-(m-1)}\nu^{2q-2m}\varepsilon_q^\nu, \qquad (15)$$

where

$$z_j = \frac{1}{2^{m-1}\nu^{2m}} x_j + \left(1 - \frac{1}{2^{m-1}\nu^{2m}}\right) y_j \in R_j \quad \text{for all} \quad j \neq m.$$

Moreover, since $s_m \in A_m = R_m \cap G_m$, there exists $(t_j)_{j\neq m} \in \prod_{j\neq m} R_j$ such that

$$s_m = -\sum_{j\neq m} t_j + \sum_{k\in I} \omega_k. \tag{16}$$

Multiplying (16) by $(1 - (1/\nu)^2)$, we have

$$\left(1 - \frac{1}{\nu^2}\right) s_m = -\sum_{j\neq m} \left(1 - \frac{1}{\nu^2}\right) t_j + \left(1 - \frac{1}{\nu^2}\right) \sum_{k\in I} \omega_k.$$

Adding this equation (and $0 = -(1/\nu^2)\sum_{h\in I_{out}}(\omega_h - \omega_h)$) to (15) and rearranging it, we obtain

$$2(s_m + \varepsilon_m^\nu) = -\sum_{j\neq m} a_j + 2\sum_{k\in I} \omega_k, \tag{17}$$

where for $j \in I_{out}$,

$$a_j = z_j + \left[\frac{1}{\nu^2}\omega_j + \left(1 - \frac{1}{\nu^2}\right) t_j\right],$$

for $j \in I_{bd} \setminus \{1, \cdots, m-1\}$,

$$a_j = z_j + \left[\frac{1}{\nu^2} s_j + \left(1 - \frac{1}{\nu^2}\right) t_j\right],$$

and for $j \in \{1, \cdots, m-1\}$,

$$a_j = z_j + \left[\frac{1}{\nu^2}\left(s_j + \frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}}\varepsilon_j^\nu\right) + \left(1 - \frac{1}{\nu^2}\right) t_j\right].$$

Note that for $j \in \{1, \cdots, m-1\}$, since

$$0 < \frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}} \leq 1,$$

we have

$$s_j + \frac{1}{2^{-[j-(m-1)]}\nu^{-(j-m)}}\varepsilon_j^\nu \in R_j.$$

Then, by (17),

$$s_m + \varepsilon_m^\nu = -\sum_{j\neq m} \frac{1}{2} a_j + \sum_{k\in I} \omega_k.$$

Since $(1/2)a_j \in R_j$ for all $j \neq m$, we have $s_m + \varepsilon_m^\nu \in A_m$, which is a contradiction. □

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