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Farsighted coalitional stability of price leadership cartel

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Abstract

This paper analyzes the farsighted behavior of firms that form a dominant price leadership cartel. We consider stability concepts such as the farsighted core, the farsighted stable sets, and the largest consistent set.

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1 Introduction

Stability of collusive behavior has been one of the staple concerns in oligopoly theories. As reported by Markham (1951) about the American industry in the first half of the twentieth century, collusion is often realized in the way that firms in a leader position announce some price and other firms follow it. Since the followers free ride on the profit-enhancing effort of the leaders and enjoy higher profits than the leaders, it is traditionally considered to be difficult for the firms to achieve and sustain collusion. Therefore the purpose of this paper is to revisit this problem from a recent development of game theoretic notions on farsighted coalitional stability.

D'Aspremont, Jacquemin, Gabszewicz, and Weymark (1983) present a model of a dominant cartel that faces a competitive fringe, where the cartel acts as a Stackerberg leader with respect to price-taking fringe firms, and they examine through stability consideration how collusive pricing prevails in the market. In d'Aspremont et al. (1983), a certain size of the cartel is "stable" if (i) no firm in the existing cartel finds it profitable to exit from the cartel and (ii) no firm outside the cartel can become better off by entering the existing cartel. So, the stability consideration by d'Aspremont et al. (1983) has a traditional problem of "myopia," which is well expressed by Fisher (1898, p. 126) in his comment on the Cournot assumption. Moreover, Diamantoudi (2005) argues

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that myopic stability notion of d'Aspremont et al. (1983) is inconsistent with the "far-sightedness" of firms embedded in price leadership model itself, and reconsiders the stability of the dominant cartel.[1]

In this paper, we follow Diamantoudi's argument and consider farsighted coalitional stability. Thus, we allow coalitional moves of players, and this is one of the different points from Diamantoudi (2005). Moreover, we consider three notions of farsighted coalitional stability because farsightedness of players can be formulated in several ways. We first examine the farsighted core, which is presented in Jordan (2006) on pillage games and Page and Wooders (2005) on network formation games. However, the farsighted core often exhibit the too-excluding-ness property because some outcome outside the farsighted core can be excluded due to the domination by the "unstable" outcome. So we need more elaborate notions of farsighted stability. A farsighted stable set that is a von Neumann and Morgenstern (1953) solution according to the indirect dominance relation and the largest consistent set introduced by Chwe (1994) are such stability concepts. Due to the theory of social situations of Greenberg (1990), these two notions are different in the expectation of the deviant players to the responses of other players: optimistic perspective constitutes the farsighted stable set and the conservative perspective constitutes the (largest) consistent set. We show that (i) the farsighted core is either an empty set or a singleton set of the grand cartel, (ii) any Pareto efficient cartel is itself a farsighted stable set, and (iii) the set of cartels in which fringe firms enjoy higher profits than the firms in the minimal Pareto efficient cartel is the largest consistent set.

The rest of this paper is organized as follows: In the next section, we present a price leadership cartel model. In Section 3, we provide a cartel formation game derived from the price leadership cartel model, and in Section 4 we provide our stability concepts of the farsighted core, the farsighted stable set, and the largest consistent set, and characterize them of the price leadership model. Section 5 provides the proof of the theorems. Section 6 is a conclusion.

2 Price Leadership Cartel

We consider an industry composed of n ($n \geqq 2$) identical firms, which produce a homogeneous output. If k firms decide to form a cartel and set a price p, the remaining $n-k$ firms constitute a competitive fringe and decide each output $q_f(p)$ by

$$p = c'(q_f(p)),$$

where c' is firm i's marginal cost, which satisfies $c(0) = 0$, $c' > 0$, and $c'' > 0$.

Let $d(p)$ be a market demand function satisfying $d(p) > 0$ and $d' < 0$. Members of a dominant cartel choose the price that maximizes their joint profit, given the supply decision of a competitive fringe. Since the marginal cost is increasing, the maximization of the joint profit is achieved by equal division of their total output. Therefore, each firm of a cartel behaves as a monopolist with respect to the individual residual demand function defined as $r(p,k) = \frac{d(p)-(n-k)q_f(p)}{k}$. Thus, the cartel chooses the

[1]Diamantoudi (2005) uses the notion of farsighted stability and shows that there exists a unique set of stable cartels. However, she does not reveal the shape and characteristic of stable cartels. This is because her existence and uniqueness theorem relies on the theorem of von Neumann and Morgenstern (1953). Nakanishi and Kamijo (2008) prove the same statements using constructive approach and characterize the stable cartels using some algorithms.

price that maximizes the profit:

$$\pi_c = r(p,k)\,p - c\,(r(p,k))\,. \tag{1}$$

According to the price as a solution of the above maximization problem, the profits of a cartel firm and a fringe firm are obtained for each cartel size k $(k = 1,\ldots,n)$ and are denoted by $\pi_c^*(k)$ and $\pi_f^*(k)$, respectively. If there is no cartel (i.e., $k = 0$), then it is assumed that the market structure is competitive. Therefore, $\pi_f^*(0)$ is defined by a profit of a fringe firm for a competitive price p^{comp}, which satisfies $d(p^{comp}) = nq_f(p^{comp})$. The conditions on the demand and cost functions guarantee $\pi_f^*(0) > 0$.

In this setting, the following holds.

Proposition 1 (d'Aspremont et al. 1983)**.** *The following properties about the profits of a cartel firm and a fringe firm hold:*

(i) $\pi_c^*(k)$ *is an increasing function in* k. **—[the size monotonicity of π_c^*]**

(ii) $\pi_f^*(k) > \pi_c^*(k)$ *for each* $k = 1,\ldots,n-1$.

The first property says that the profit of each cartel firm increases as the cartel size increases. The second property of Proposition 1 says that the profit of a cartel member is less than the profit of an associated fringe members.

The next proposition shows that cartel and fringe firms prefer a situation with a dominant cartel of any size to one without it.

Proposition 2. *The followings hold:*

(i) $\pi_c^*(k) > \pi_f^*(0)$ *for* $k = 1,\ldots,n$. **—[the cartel desirability for π_c^*]**

(ii) $\pi_f^*(k) > \pi_f^*(0)$ *for* $k = 1,\ldots,n-1$. **—[the cartel desirability for π_f^*]**.

Proof. Suppose that there exists a size k cartel. If the cartel chooses price $p = p^{comp}$, then they can gain $\pi_f^*(0)$, the profit at a competitive equilibrium. Since cartel members set a price to maximize their profit, their maximized profit must be equal to or greater than $\pi_f^*(0)$. So it suffices to show the strict inequality in the case of $k = 1$. The first order condition for maximizing (1) yields

$$d(p) - (n-1)q_f(p) + (p - c')(d' - (n-1)q_f') = 0.$$

When $p = p^{comp}$, $r(p,1) = q_f(p)$ and so $p = c'(r(p,1))$. Thus, we have $d(p^{comp}) - (n-1)q_f(p^{comp}) = q_f(p^{comp}) = 0$. This means $\pi_f^*(0) = 0$; this is contradiction.

The second statement of this proposition follows from (i) and Proposition 1-(ii). □

In the following sections, we consider only the four properties described in Propositions 1 and 2, instead of considering how π_c^* and π_f^* are determined. So, a scope of our discussion is any kind of coalition formation satisfying these conditions. These conditions are quite general, but are different from Thoron (1998) who analyzes cartel formation in general setting. The size monotonicity of π_c^* and the cartel desirability for π_c^* are more demanding conditions than her requirement of an oligopoly game being essential. Proposition 1-(ii), the free riding of fringe firms, is almost equivalent to her condition A_2. Her condition A_1 that is the size monotonicity of π_f^* is not required in our model.

3 Cartel formation game

We describe a problem of cartel formation as the following normal form game. Let $N = \{1, \ldots, n\}$ be a set of players (firms). Each firm i faces a binary choice problem of joining or not joining a cartel in formation of a cartel. Thus, its action is either $x_i = 1$ (joining a cartel) or $x_i = 0$ (not joining a cartel), where $x_i \in X_i = \{0, 1\}$. Thus, $x = (x_i)_{i \in N} \in X = \times_{i \in N} X_i$ is a strategy profile of firms and at the same time it represents a cartel specified by firms' decision. We often see each x as an outcome of the cartel formation game. The set of members in cartel x is denoted by $C(x) = \{i \in N : \ x_i = 1\}$ and its size is denoted by $|x| = |C(x)| = \sum_{i \in N} x_i$ for convenience. The set of fringe firms is denoted by $F(x) = N \setminus C(x)$. Given $x, y \in X$, $x \wedge y$ denotes a cartel z such that $z_i = \min\{x_i, y_i\}$ for $i = 1, \ldots, n$. We can easily verify that $C(x \wedge y) = C(x) \cap C(y)$.

For each $i \in N$, the payoff function $u_i : X \to \mathbb{R}$ is defined by

$$u_i(x) = \begin{cases} \pi_c^*(|x|) & \text{if } x_i = 1, \\ \pi_f^*(|x|) & \text{if } x_i = 0. \end{cases}$$

Thus, the payoff of a firm depends only on the size of the current cartel and whether it belongs to the cartel or not.

A cartel formation game is described by $G^{cf} = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$.

In the rest of this section, we explain additional notations used in this paper. For a coalition $S \subseteq N$, we write $x \succ_S y$ if $u_i(x) > u_i(y)$ holds for any $i \in S$. Let $x^c \in X$ and $x^f \in X$ denote $(1, \ldots, 1)$ and $(0, \ldots, 0)$, respectively. That is, x^c represents the grand cartel and x^f represents a competitive situation. Let $x \in X$ and $y \in X$ be two distinct cartels. We say that a cartel x *Pareto dominates* y, and denote xPy if $u_i(x) \geqq u_i(y)$ holds for all $i \in N$ and strict inequality holds for some $j \in N$. If x is not Pareto dominated by any other cartel, the x is called a *Pareto efficient* cartel. The set of all the Pareto efficient cartels is denoted by $X^P \subseteq X$. Since the grand cartel $x^c = (1, \ldots, 1)$ is Pareto efficient by the size monotonicity of π_c^* and the cartel desirability of π_c^*, X^P is not empty. On the other hand, since $x^c P x^f$ by the cartel desirability of π_c^*, $x^f \notin X^P$. Next we say that $x \in X$ is individually rational if for all $i \in N$, $u_i(x) \geqq v_i := \min_{y_{-i}} \max_{y_i} u_i(y)$. In our model, from Propositions 1 and 2, $v_i = \pi_c^*(1)$. Thus, the set of individually rational cartels, X^I, is $X^I = X \setminus \{x^f\}$.

4 Farsighted coalitional stability

In this section, we consider which cartels are stable in the cartel formation game defined in the previous section, taking into account the farsightedness of players (firms). To analyse the stable outcome, we first define the inducement relations. The inducement relations $\{\to_S\}_{S \subseteq N}$ specify what firms can do in a formation of a cartel. $x \to_S y$ means that coalition S can change a cartel from an initial cartel x to another y. Under the free entry and exit assumption, we have $x \to_S y$ if and only if $x_i = y_i$ holds for any $i \in N \setminus S$. Note that $x \to_S y$ means neither that coalition S can enforce y no matter what any other firm does nor that S must move y from x whenever x is a status quo.

The situation we assume is described as the one considered Chwe (1994). At any given time, there is a status quo cartel, say x, which can be changed to another cartel y if the firms of S decide to change x to y and $x \to_S y$, and from this new status quo y, other coalitions might move, and so forth. The moves by some coalition are observed by all the players and they know what the status quo is. If a status quo cartel z is reached and no coalition decides to move from z, then z is called a "stable" outcome and the game is over; then the firms receive their payoff from z.

Next we explain the dominance relations over X. A cartel x is directly dominated by y via coalition S if $x \to_S y$ and $y \succ_S x$. If there is some S such that x is directly dominated by y via S, we write $y > x$. The logic behind the core, the set of cartels that are not dominated by any other cartel, is that if coalition S can change a cartel from x to y and all the members in S prefer y to x, then x is not stable; if x is not dominated by any other cartel, it is stable. The stability notion established in the core is based on the presumption of individuals with myopic perspective when they contemplate to move or not because they do not anticipate other coalition's reaction against their move. If they fully take into account the further deviation of other coalitions subsequent to their own deviation, they may find that they would become better off in the final outcome that is realized after reactions of the other coalitions.

A dominance relation that is extended to describe such kinds of anticipation is indirected dominance relation considered in Chwe (1994) that captures the ability for each firm to foresee the final outcome which is induced by the firms' current move.[2]

Definition 1. *A cartel x is* ***indirectly dominated*** *by y and we write $y \gg x$ if there exist a finite sequence of cartels $x^0, x^1, \ldots, x^m$ with $x^0 = x$ and $x^m = y$ and a sequence of coalitions $S^1, \ldots, S^m$ such that for each i $(= 1, \ldots, m)$,*

(i) $x^{i-1} \to_{S^i} x^i$, and

(ii) $y = x^m \succ_{S^i} x^{i-1}$.

Thus, the indirect dominance relations allow coalitions to look arbitrary many steps ahead. In contrast, the direct dominance relations, by setting $m = 1$ in the above definition, entails that coalitions look only at the next step.

One remark on the indirect dominance is that the agreements on the deviation by firms are not binding, and they always consider a further deviation of the other firms.

With indirect dominance relations, the idea of the core is extended to the farsighted one, which is considered by Jordan (2006) on pillage games and by Page and Wooders (2005) on network formation games.

Definition 2. *A set $C \subseteq X$ is called a* ***farsighted core*** *if $x \in C$ is not indirectly dominated by any other cartel $y \in X$.*

The farsighted core is the set of cartels that are not indirectly dominated by any other cartel. So, cartels in the farsighted core are stable in a sense that once a cartel in the farsighted core is realized, then further deviation cannot occur even though individuals consider a chain of responses of the other individuals caused by their first deviation. By its definition, the farsighted core is the subset of the core which is the set of the cartels that are not directly dominated by any other cartels.

The farsighted core of the price leadership model is characterized as follows.

Theorem 1. *If there is no cartel $x \in X$ such that $\pi_f^*(|x|) > \pi_c^*(n)$, the farsighted core is $\{x^c\}$. Otherwise, the farsighted core is empty.*

[2]Harsanyi (1974) first consider the indirect domination in order to repair the inconsistency in the definition of the stable set in a characteristic function form game. However, his definition is slightly different from one considered by Chwe (1994).

Thus, in some situation, the farsighted core does not give any prediction. This may support the following argument on the too-excluding-ness of the farsighted core. The farsighted core can be too excluding because some element outside the farsighted core could be indirectly dominated by a cartel outside the farsighted core. This means that some cartel may be considered to be "unstable" because it is indirectly dominated by another "unstable" cartel. In this meaning, players presumed in the farsighted core do not take care the "stableness" of their prospective outcome when they decide to move.

A stability concept that reflects such kind of farsightedness of players is a farsighted stable set, which is a stable set or von Neumann and Morgenstern (1953) solution according to the indirect dominance relations.

Definition 3. *A subset K of X is called a **farsighted stable set (FSS)** if the following conditions hold:*

(i) For any $x \in K$, there does not exist $y \in K$ such that $y \gg x$ (internal stability of K).

(ii) For any $z \in X \setminus K$, there exists $x \in K$ such that $x \gg z$ (external stability of K).

The internal stability implies that between any two outcomes in the farsighted stable set, there is no group of players whose members all prefer one to others and achieve preferred outcome. External stability implies that for any outcome outside the set, there is a group of players whose members have a commonly preferred outcome in the set and can realize it through the chain of deviation starting with their own deviation.

A farsighted stable set presumes the following standard of behaviors of individuals. Suppose that outcomes in set K is commonly considered to be "stable" and outcomes outside K to be "unstable" by all the individuals. Then, once an outcome x in K is reached, any deviation from x never occurs because there exists no stable outcome that indirectly dominates x, and if in time an outcome y outside K is reached, there exists stable outcome $x \in K$ that indirectly dominates y.

The next theorem states that any Pareto efficient cartel constitutes a farsighted stable set.

Theorem 2. *For any $x \in X^P$, $\{x\}$ is an FSS.*

Moreover, next theorem shows that there is no farsighted stable set other than the ones defined in Theorem 2.

Theorem 3. *There exists no FSS other than that given in Theorem 2.*

From Lemma 1 (see Section 4), $x \in X \setminus \{x^c\}$ is Pareto efficient if and only if the fringe firms in x enjoys more profit than the profit of the cartel firm in x^c. Thus, from Theorems 1, 2, and 3, if $\{x^c\}$ is the farsighted core, it is also the unique farsighted stable set.

Similar farsighted stability concept is a consistent set. A consistent set L offers a consistent treatment to judging whether some outcome is stable or not and is defined by the condition on a set of outcomes not the condition on an individual outcome. A consistent set L is such that for any one step deviation by S from a "stable" outcome x in L to another y, there exists a "stable" outcome z in L that deters the deviation of S; z is equal to y or it indirectly dominates y, and at least one player in S can not better off in z.

Definition 4. *The set $L \subseteq X$ is **consistent** if $x \in L$ if and only if for any S, y with $x \to_S y$, there exists $z \in L$ with $z = y$ or $z \gg y$ such that $z \not\succ_S x$.*

Farsighted stable sets can be re-defined by similar consistent story on the stability of a set of outcomes. A set K is a farsighted stable set if $x \in K$ if and only if for any S, y with $x \rightarrow_S y$, and for any $z \in K$ with $z = y$ or $z \gg y$, $z \not\succ_S x$. So, the difference in the two notions is two polar type of expectations to the other players' reaction by the deviant players. In consistent sets, players who are contemplating deviation refrain from deviant move when they find that they are not better off in "one" of the possible outcomes after their deviation; in this sense, players have a pessimistic or conservative perspective. On the other side, in FSSs, players considering deviation refrain from deviation when they find that they are not better off in "all" of the possible outcomes after their deviation; in this sense, players have an optimistic perspective. In fact, these two notions correspond to two polar kinds of stable standard of behavior, an optimistic stable standard of behavior and a conservative stable standard of behavior, in the theory of social situations of Greenberg (1990). For a detailed discussion, see Chwe (1994) and Section 3 of Suzuki and Muto (2005).

As well as farsighted stable sets, a uniqueness problem may arise when we apply consistent set. However, it has a nice property that the union of all the consistent sets is also consistent, and thus, the **largest consistent set (LCS)** is uniquely determined (Proposition 1 of Chwe, 1994). Moreover, the nonemptiness of the LCS for the finite case is guaranteed (Proposition 2 of Chwe, 1994). For other condition of non-emptiness of the LCS, see Xue (1997). Unfortunately, our price leadership mode does not satisfy these conditions. However, as in the next theorem, we show that the LCS is non-empty and is characterized by the set of cartels wherein the fringe firms enjoy more profit than that of the cartel firm in the minimal Pareto efficient cartel.

Let k^* be a minimal size of cartels that are Pareto efficient. Thus, $k^* = \min\{|x| : x \in X^P\}$.

Theorem 4. *The LCS for the price leadership model is*

$$L^* = \{x \in X : \pi_f^*(|x|) \geqq \pi_c^*(k^*)\} \cup \{x^c\}.$$

Since $X^P = \{x \in X \setminus \{x^c\} : \pi_f^*(|x|) > \pi^*(n)\} \cup \{x^c\}$ from Lemma 1, this theorem implies that $X^P \subseteq L^*$. Moreover, by the cartel desirability for π_c^*, x^f is not included in L^*. Thus, $L^* \subseteq X^I$. Consequently, we have $X^P \subseteq L^* \subseteq X^I$.

The farsighted coalitional stability considered in this paper can be applied for any any normal form game. Thus, the comparison of our results with other type of games is useful. Suzuki and Muto (2005) analyse the farsighted coalitional stability in n person prisoners' dilemma game $G^{pd} = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, ..., n\}$, $X_i = \{0, 1\}$ and $u_i : X \rightarrow \mathbb{R}$ is such that $u_i(x) = f(|x|)$ if $x_i = 1$ and $u_i(x) = g(|x|)$ if $x_i = 0$ where $f : \{1, \ldots, n\} \rightarrow \mathbb{R}$ and $g : \{0, \ldots, n-1\} \rightarrow \mathbb{R}$ satisfy the following:

(p1) $g(k-1) > f(k)$ for all $k = 1, .., n$;

(p2) $f(n) > g(0)$;

(p3) $f(k)$ is increasing in k;

(p4) $g(k)$ is increasing in k.

In this setting, they show that a Pareto efficient and individual rational outcome is a farsighted stable set in n person prisoners' dilemma game and except for degenerate cases, there does not exist other type of farsighted stable sets. Moreover, they show that under some conditions, the largest consistent set is the set of all the individually rational outcomes, which means that the largest consistent set includes the Pareto inefficient outcomes. In this sense, these results are similar to our Theorem 2 and 4. On the other hand, the largest consistent set in our model can be the strict subset of X^I. This difference between the price leadership model and n person prisoners' dilemma comes from the cartel desirability in our model, which leads to the too large set of individually rational outcomes. The farsighed core is always empty in the case of n person prisoners' dilemma game, in contrast to our Theorem 1, which implies that only the grand cartel can be in the farsighed core under some condition. This is because the condition (p1) in the n person prisoners' dilemma is weaken to Proposition 1-(ii) in the price leadership model.

5 Proofs of the theorems

We first present the following lemma that chatecterize the set of the Pareto efficient cartels.

Lemma 1. *The following three statements on $x \neq x^c$ are equivalent:*

(a) $x \in X^P$.

(b) $\pi_f^*(|x|) > \pi_c^*(n)$.

(c) $x \gg x^c$.

Proof. (a) → (b). Suppose that $x \in X^P$, $x \neq x^c$ but $\pi_f^*(|x|) \leqq \pi_c^*(n)$. Then x^c Pareto dominates x by the size monotonicity of π_c^* and this contradicts $x \in X^P$.

(b) → (a). Suppose that x satisfies $\pi_f^*(|x|) > \pi_c^*(n)$ and that there exists $y \in X$ such that yPx. For $i \in F(x)$, y_i must be '0' since $u_i(x) = \pi_f^*(|x|) > \pi_c^*(n) \geqq \pi_c^*(|y|)$, where the last inequality follows from the size monotonicity of π_c^*. Thus, $C(x) \supsetneq C(y)$. Moreover, $C(y) \neq \emptyset$ because of the cartel desirability for π_c^*. So, there exists $i \in C(y) \subsetneq C(x)$. For this i, the size monotonicity of π_c^* implies $u_i(y) = \pi_c^*(|y|) < \pi_c^*(|x|) = u_i(x)$. This contradicts yPx.

(a) → (c). Suppose $x \in X^P \setminus \{x^c\}$. Then, by (a) → (b) of this lemma, $u_i(x) > u_i(x^c)$ for any $i \in F(x)$. Thus, x directly dominates x^c via members in $F(x)$ simultaneously exiting the cartel.

(c) → (a). If $x \gg x^c$ holds, then there exists the first deviant coalition S from x^c to the final outcome x. Hence, $x \succ_S x^c$ holds. Let $i \in S$. Then, $x_i = 0$ because there is no cartel better for a cartel firm than x^c. The fact that $u_i(x) > u_i(x^c)$ implies that $\pi_f^*(|x|) > \pi_c^*(n)$. Thus, x is Pareto efficient by (b) → (a) of this lemma. □

The following lemma gives a sufficient condition on the indirect dominance relations.

Lemma 2. *For any two distinct cartels $x, y \in X$ with $x \neq x^c$ and $y \neq x^f$, $x \gg y$ if $\pi_f^*(|x|) > \pi_c^*(|y|)$.*

Proof. Take any $x, y \in X, x \neq y, x \neq x^c, y \neq x^f$ such that $\pi_f^*(|x|) > \pi_c^*(|y|)$. We show that $x \gg y$ by constructing a sequence of moves that is starting with y and ending with x, where every acting coalition prefers x to the status quo. We separete the three cases according to the inclusion relation between $C(x)$ and $C(y)$: (i) $C(y) \setminus C(x) = \emptyset$, (ii) $\emptyset \neq C(y) \setminus C(x) \subsetneq C(y)$, and (iii) $C(y) \setminus C(x) = C(y)$.

(i). $C(y)$ is included in $C(x)$. Thus, $|x| > |y|$ holds. Consider the following sequence of two step move:

$$y \to_{C(y)} x^f \to_{C(x)} x.$$

By the size monotonicity of π_c^* and the cartel desirability of π_c^*, for $i \in C(y)$, $u_i(y) = \pi_c^*(|y|) < \pi_c^*(|x|) = u_i(x)$, and for $i \in C(x)$, $u_i(x^f) = \pi_f^*(0) < \pi_c^*(|x|) = u_i(x)$. Thus, $x \gg y$ holds through the above dominance path.

(ii). When $C(x) \subset C(y)$, a simple one step deviation $y \to_{C(y)\setminus C(x)} x$ constitutes a dominant path because $i \in C(y) \setminus C(x)$ prefer $u_i(x) = \pi_f^*(|x|)$ to $u_i(y) = \pi_c^*(|y|)$ by the presumption of this lemma. When $C(x) \not\subset C(y)$, consider the following sequence of three step move:

$$y \to_{C(y)\setminus C(x)} x \wedge y \to_{C(x\wedge y)} x^f \to_{C(x)} x.$$

In the first step of deviation, $i \in C(y) \setminus C(x)$ prefers x to y because i is in the fringe position of x and $\pi_f^*(|x|) > \pi_c^*(|y|)$. In the second step, for $i \in C(x \wedge y)$, $u_i(x \wedge y) = \pi_c^*(|x \wedge y|) < \pi_c^*(|x|) = u_i(x)$ holds because $|x \wedge y| < |x|$ and π_c^* satisfies size-monotonicity. The third step follows from the cartel desirability of π_c^*.

(iii). $C(x)$ and $C(y)$ has an empty intersection. Consider the following two step deviation: $y \to_{C(y)} x^f \to_{C(x)} x$. The incentives of the firms in each step are satisfied by the similar argument to cases (i) and (ii). □

Proof of Theorem 1. We first show that $x^c \in X^P$ indirectly dominates any other cartel $y \in X \setminus \{x^c\}$ by the following dominance sequence: $y \to_{C(y)} x^f \to_N x^c$. The incentive conditions hold because in the first step $\pi_c^*(|y|) < \pi_c^*(n)$ by the size monotonicity of π_c^* and in the second step $\pi_f^*(0) < \pi_c^*(n)$ by the cartel desirability for π_c^*. So, the farsighted core is either $\{x^c\}$ or an empty set. Suppose that $\pi_f^*(|x|) \leq \pi_c^*(n)$ holds for any $x \in X \setminus \{x^c\}$. Then, by Lemma 1, there is no x such that x indirectly dominates x^c. Thus, the farsighted core is $\{x^c\}$. On the other hand, if $\pi_f^*(|x|) > \pi_c^*(n)$ for some $x \in X$, by Lemma 1, this x indirectly dominates x^c. Thus, the farsighted core is empty. □

Proof of Theorem 2. Since $\{x\}$ consists of one point, we only consider the external stability. To show $x \in X^P \setminus \{x^c\}$ indirectly dominates any other cartel is immediate consequence from Lemmas 1 and 2. We know from Lemma 1 that $x \in X^P \setminus \{x^c\}$ satisfies $\pi_f^*(|x|) > \pi_c^*(n) \geqq \pi_c^*(|y|)$ for any $y \in X \setminus \{x^f\}$, and from Lemma 2 that $x \gg y$ holds if $\pi_f^*(|x|) > \pi_c^*(|y|)$ holds; thus this x indirectly dominates any $y \neq x^f$. When $y = x^f$, x dominates y by a simple dominance path $x^f \to_{C(x)} x$ because all the firms in x prefer x to x^f by the cartel desirability of π_c^*.

On the other hand, $x^c \in X^P$ also indirectly dominates any other cartel $y \in X\setminus\{x^c\}$ as shown in the proof of Theorem 1. □

Proof of Theorem 3. Take any farsighted stable set K that is different from the set given in Theorem 2. Then, K does not contain any outcome that belongs to X^P since, as shown in Theorem 2, such a cartel indirectly dominates all the others and this contradicts the internal stability of K. So, if $x \in K$, then $x \notin X^P$. Thus $K \cap X^P = \emptyset$ and this implies that by Lemma 1 there exists no $x \in K$ such that $x \gg x^c$. Since $x^c \in X^P$ and thus $x^c \notin K$, the external stability of K does not hold. □

Proof of Theorem 4. We first cite the following proposition of Chwe (1994).

Proposition 3 (Chwe, 1994). *If $K \subseteq X$ is an FSS, K is a subset of the LCS.*

Then, Theorem 2 together with Proposition 3 implies $X^P \subseteq$ LCS. The proof proceeds with first showing an element in $X \setminus L^*$ does not belong to LCS, and then showing that the set described in the theorem is actually a consistent set.

Take any $x \in X$ with $\pi_f^*(|x|) < \pi_c^*(k^*)$. Consider a deviation from x to x^c by $F(x)$, thus, $x \to_{F(x)} x^c$. For any $y \in$ LCS satisfying $y \gg x^c$ or $y = x^c$, this y must belong to X^P by Lemma 1. Thus, by Lemma 1 and the size monotonicity of π_c^*,

$$\pi_f^*(|y|) > \pi_c^*(n) \geq \pi_c^*(k^*) > \pi_f^*(|x|).$$

This implies $y \succ_{F(x)} x$. This means that x does not belong to LCS.

Next, we show that L^* is a consistent set.

[if part] We will show that for all $x \notin L^*$, there exist $y \in X$ and $S \subseteq N$ with $x \to_S y$ such that for all $z \in L^*$ with $[z = y$ or $z \gg y]$ and $z \succ_S y$. Take any $x \notin L^*$. Again, consider a deviation from x to x^c by $F(x)$. Then, for any y that indirectly dominates x^c, all the firms in $F(x)$ prefers y to x. Thus, we have the desired result.

[only if part] We will show that for all $x \in L^*$, all $y \in X$ and all $S \subseteq N$ such that $x \to_S y$, there exists $z \in L^*$ such that $[z = y$ or $z \gg y]$ and $z \not\succ_S y$. Take any $x \in L^*$ and any S, y such that $x \to_S y$. If $S \cap F(x)$ is not empty, there exists $i \in S \cap F(x)$. By Theorem 2, any element in X^P indirectly dominates any other cartel. Thus, $z \in X^P \subseteq L^*$ with $|z| = k^*$ and $i \in C(z)$ also indirectly dominates x. By definition of L^*, $u_i(x) = \pi_f^*(|x|) \geqq \pi_c^*(k^*) = \pi_c^*(|z|) = u_i(z)$. So, $z \succ_S x$ does not hold. On the other hand, if $S \cap F(x)$ is empty, $C(y) \subset C(x)$ and, thus, $|y| < |x|$. By the size monotonicity of π_c^* and Proposition 1-(ii), we have

$$\pi_f^*(|x|) > \pi_c^*(|x|) > \pi_c^*(|y|),$$

and by Lemma 2, $x \gg y$ holds. However, $x \succ_S x$ does not hold.

Thus, L^* is consistent and it is largest because addition of any outcome outside L^* breaks the consistency of L^*. □

6 Conclusion

In this paper, we analyzed the stability of a dominant cartel of the price leadership model introduced by d'Aspremont et al. (1983). The stability concepts adopted in this study are the farsighted core, the farsighted stable set, and the largest consistent set.

We found the complete shapes of farsighted stable sets, the largest consistent set, and the farisighted core. Our results imply the possibility of cooperation in the dilemma situation and shed some light on dissolution of the dilemma, which has been widely studied by non-cooperative approach and equilibrium concept. In addition, our discussions have policy implications on a market structure because our theorems show that there is the possibility of firms forming a large cartel even if the decision problem of the firms joining or not joining a cartel is in a dilemma situation. These implications are in contrast to the results of Selten (1973) and Prokop (1999) who analyze cartel formation in non-cooperative way and show that firms encounter some difficulty to form a large cartel.

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